UNIVERSITY OF TORONTO
3 1761 00028651 8

ELASTICITY AND STRENGTH

OF

MATERIALS USED IN ENGINEERING CONSTRUCTION

SECTION III

Theory of Torsion in Shafting
AND
Double Bending of Plates

BY

186121.
19.12.23

C. A. P. TURNER, Consulting Engineer
MINNEAPOLIS, NEW YORK, WINNIPEG
NEW ORLEANS

MEM. AM. SOC. M. E.
MEMBER AM. SOCIETY TESTING MATERIALS
MEMBER SOC. PROMOTION ENGINEERING EDUCATION
MEM. AM. SOC. C. E.

MINNEAPOLIS, MINN., 1923

PRICE, $5.00
COPYRIGHT, 1923
ALL RIGHTS RESERVED

DEDICATION

TO THE MEMORY OF J. M. TURNER, FAMILIARLY KNOWN AS UNCLE JOHN TO HIS COMRADES AND MEN OF THE 2ND R. I. VOLUNTEER REGIMENT OF '61, THIS WORK IS DEDICATED AS AN APPRECIATION OF THE VALUE OF HIS INSTRUCTION GIVEN HIS SON FORTY-FIVE YEARS AGO, IN THE SCIENCE OF SHIP DRAFTING, THE TREATMENT OF CANT FRAMES FOLLOWING GRIFFITHS AND THE PROPERTIES OF FAIR CURVES AS DEVELOPED BY THE METHODS THEN IN USE.

TA
405
T8
v.3

PREFACE

This section carries the graphic analysis of states of stress forward to the determination of contours of bulge and depression in the shafting applying the elementary principles of Section I, which shows that displacement curves bisect the angle between the curve of greatest shear and principal stress. By this method the planes of rupture are readily located and rupture itself is found to be caused by the joint effect of shear and principal stress.

Graphic analysis gives a clear insight into the flexure of plates, disclosing that the squeeze or stretch of the neutral plane of the plate is an important element in its resistance heretofore disregarded in mathematical analysis. The reverse sign of the twist of a vertical element above and below the neutral plane renders futile the analysts' endeavor to correlate the phenomena of plate action with the coefficient K of lateral contraction under pure tensile stress. The curves of greatest shear make an angle of 45° to the direction of the principal pure stress in developing the deformation represented by the coefficient K whereas in plate action wide variation of angularity of the curves of greatest shear with the lines of principal radial stress occurs demonstrating the unrelated nature of the phenomena.

It is thought that the new method of investigation will give the student that concise conception of the relation of states of stress to deformation involved in the problems under consideration so difficult to convey by the involved equations of the mathematical theory of elastic solids and so in time tend to eliminate fundamental errors in the rules of design in building code laws.

The reader is indebted almost entirely for Chapter XVI to the work of the author's co-worker, Dr. H. T. Eddy, and for his joint work on portions of Chapters XIII and XIV. See Concrete Steel Construction. Had Dr. Eddy been spared to co-operate in the presentation of stress line analysis as he was planning to do just before his death the matter would have been presented in a better manner by aid of his superior mathematical ability.

C. A. P. TURNER

TABLE OF CONTENTS

CHAPTER XII

ANALYSIS OF ROUND SHAFTS

1. Introduction. 2. Curves of Greatest Shear and Principal Stress. 3. Reverse Summation of Tension and Compression in Torsion. 4. Comparative Twisting and Bending Resistance. 5. Computation of Angular Distortion. 6. Diameter of a Round Shaft Required to Resist a Given Twisting Moment. 7. Diameter Required to Transmit a Given Number of Horse Power at a Given Speed. 8. The Resultant Principal Stress from Torsion and Flexure. 9. Limitation of Practical Formulas. 10. Simplified Computation. 11. Regulation of Variable Torque. 12. Limiting Deflection. 13. Summary.

Pages 1 to 13.

CHAPTER XIII

TORSIONAL ANALYSIS OF SQUARE RECTANGULAR TRIANGULAR AND OVAL PRISMS

1. Warping of the Normal Section. 2. Curves of Greatest Shear. 3. Curves of Principal Stress. 4. Relative Rigidity of Square and Round Shafts. 5. Strength of Rectangular Prisms Under Torsion. 7. Recapitulation and Remarks.

Pages 14 to 21.

CHAPTER XIV

ANALYSIS OF COMBINED BENDING AND TWISTING RESISTANCE IN HOMOGENEOUS FLAT PLATES

1. Characteristics of Plate Action Developed. 2. Separation of Bending and Twisting in Plate Action. 3. Circumferential and Radial Bending Resistances. 4. Phenomena of Buckling in Thin Plates. 5. Stress Curves and Displacement. 6. Separation of States of Surface Stress and Curvature of Uniform Plates into Simple Components. 7. Separation and Composition of Curvatures. 8. The Sum of the Reciprocals of the Radii of Curvature on any Orthogonal Pair of Planes Normal to the Surface of the Uniform Elastically Bent Plate a Constant. 9. Planes of Principal Curvature at any Point of a Uniform Bent Plate Orthogonal to Each Other. 10. Twisting Shears Change Sign in Passing Planes of Principal Curvature. 11. Effective Moment—Interference of Moments of Like and Unlike Sign. 12. Applied and Resisting Moments of Square Panels Supported on Four Sides. 12b. Applied Bending and Twisting Moments about the Diameter of a Circular Plate Supported at its Center. 13. Statical Demonstration of the Bending Moment Parallel to each Edge of a Square Panel of a Column Supported Plate. 14. Equilibrium of an Elementary Vertical Prism under Bending Moment and Twisting Moment. 15. Warping and Twisting of a Square Plate Supported at its Edges. 16. Stress Analysis of a Square Plate, Anticlastic Bending. 17. Stress Analysis of a Circular Plate Supported at its Periphery. 18. Stress Analysis of a Square Plate Simply Supported at its Corners. 19. Stress Analysis of a Square Panel of a Continuous Column Supported Plate Loaded at the Diagonal Center of all Panels. 20. Loci of the Cusp of Curves of Displacement. 21. Analysis of Square Plates: 22. Poisson Effect. 23. Relative Magnitude of Twisting and Bending Couples. 24. Strength of Plates in Double Curvature. 25. Summary

Pages 22 to 60.

CHAPTER XV

ANALYSIS OF COMPOSITE PLATES AND THE THEORY OF CONTINUITY WITH VARIABLE MOMENT OF INERTIA

1. Introductory. 2. Localization of Reinforcement Changes the Manner of Storage of Internal Work. 3. Relation of Cross Section of Reinforcement that Resists Bending in Dish Shaped Curvature to the Form of the Contour. 4. Mechanics of Embed-

ment. 5. Resistance to Deformation of the Bond Stresses of Crossed Rods. 6. Analytical Investigation of Bending Moments in a Square Panel. 7. Magnitude of the Total Bending and Twisting Moments about Horizontal Axes in a Rectangular Panel of a Uniform Unrestrained Slab. 8. Moments at Median Section and Margins of Interior Panels in Uniform Column Supported Flat Slabs Uniformly Loaded. 9. Apportionment of Moment with Variable Moments of Inertia Along Direct Spans under Uniform Load. 10. Moment of Inertia of Reinforced Flat Plates. 11. Apportionment of Moment Diagonal Spans of Square and Rectangular Panels of Column Supported Slabs. 12. Deflection in Panel of a Continuous Column Supported Flat Slab Floor Uniformly Loaded. 13. Arrangement of Reinforcement in Spiral Mushroom System. 14. Computation. 15. Steel Ratios and Minimum Thickness of Slabs. 16. Increase in Depth of Slab Required for Increase in Load. 17. Size and Spacing of Rods in Flat Slab Construction. 18. Summary.

Pages 61 to 103.

CHAPTER XVI

UNBALANCED MOMENT IN MONOLITHIC FLOORS AND COLUMNS

1. Introduction. 2. Theory of Uniform Continuous Beam. 3. Moments and Deflections in a Single Uniformly Loaded Span of a Continuous Beam. 4. Unbalanced Loads on a Floor Resting on Rectangular System of Continuous Girders Integral with Supporting Columns. 5. Flat Slab Floors. 6. Wall Panels. 7. Momentof Inertia. 8. Unbalanced Test Loads. 9. Summary.

Pages 104 to 121.

CHAPTER XII

EXACT AND APPROXIMATE ANALYSIS OF SIMPLE TORSION OR TWIST OF ROUND SHAFTS AND COMBINED TORSION AND FLEXURE THEREIN

1. Torsion occurs when a body is acted upon by two couples or moments of opposite sign, as in shafting transmitting power to machines.

As the deformations involved are complex, they may best be explained by reference to a model. Soft vulcanized rubber is ideal for this purpose, since its limit of elastic deformation is so much greater than possessed by materials of construction that the elastic deformations can be made so large that their nature and character may be observed without difficulty.

Upon a cylindrical piece of rubber, say $\frac{3}{4}$-inch in diameter and six inches long, draw parallel lines on the surface along elements a quarter inch apart. Draw normals to these around the circular section, also spaced a quarter inch apart. Clamp one end and twist the other.

By twisting, the elementary quarter inch squares have become deformed by shear into rhomboids. The circular transverse lines still remain in planes normal to the axis but the longitudinal elements or lines will be deformed into spirals, and the longitudinal diametral plane of their original position has become deformed into a ribbon spiral in which the elongation of fibers parallel to the axis increases approximately as their radial distance from the axis.

The reaction of this spiral stretching is a longitudinal compression shortening the length, greatest at the surface and for some distance below. The squeeze of the spiral twist causes radial compression on all transverse sections of intensity increasing toward the axis. This lateral compression induces longitudinal elongation and the resultant change in length is the resultant of these two opposite effects.

From the principle that volume is a kinetic invariant of elastic strain and the linear law of increase in spiral tensions, the neutral cylinder between the zones of longitudinal elongation and shortening (or tension and compression) is .7071 x R or .7071 R from the axis.

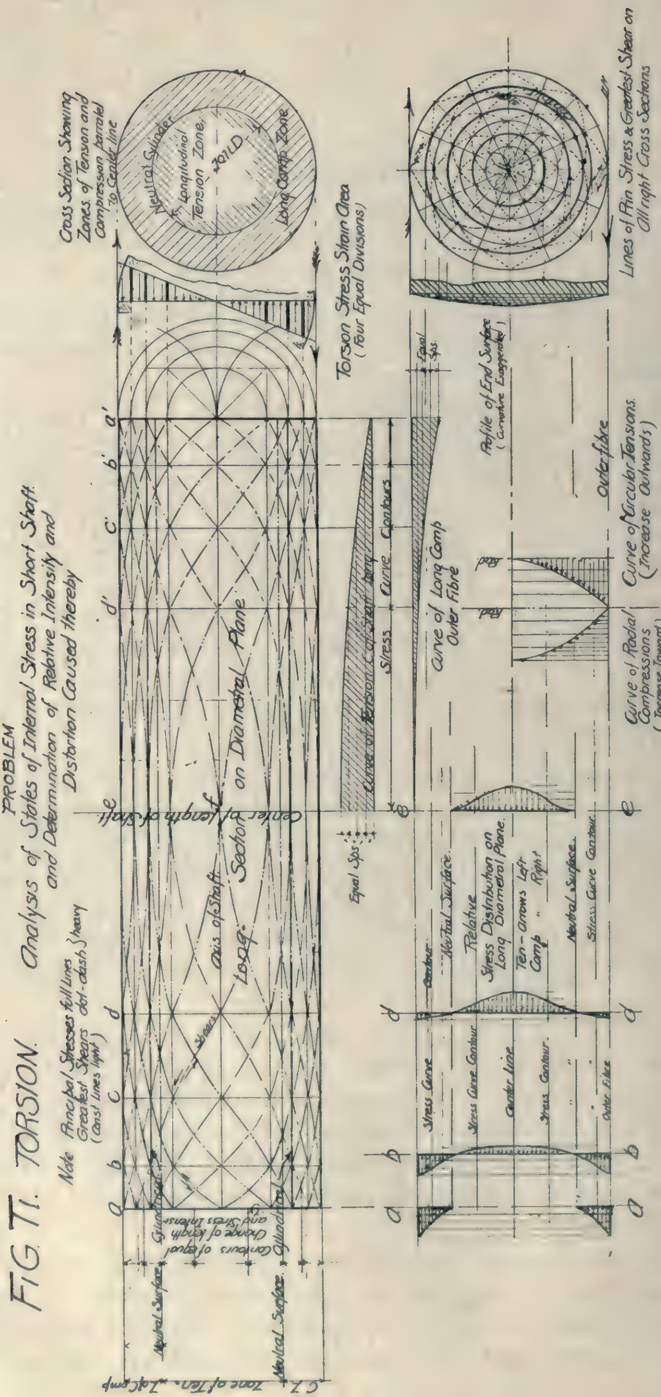


FIG T1. TORSION.
PROBLEM
Analysis of States of Internal Stress in Short Shaft and Determination of Relative Intensity and Distortion Caused thereby
Note Principal Stresses full lines Greatest Shears dot-dash Sheary (Const Lines light)
Cross Section Showing Zones of Tension and Compression parallel to Center line
Neutral Cylinder
Longitudinal Tension Zone
.707 D
Long Comp Zone
Torsion Stress Strain Area (Four Equal Divisions)
Lines of Prin Stress & Greatest Shear on All right Cross Sections
Equal Sps.
Profile of End Surface (Curvature Exaggerated)
Curve of Long Comp Outer Fibre
Outer fibre
Curve of Circular Tensions (Increase Outwards)
Curve of Radial Compressions (Increase Inward)
Rad
Rad
Stress Curve Contours
Curve of Torsion Cor Shear long
Long Section on Diametral Plane
Center of length of Shaft
Axis of Shaft
Shears
Equal Sps
Contour
Neutral Surface
Relative Stress Distribution on Long Diametral Plane
Ten-arrows Left
Comp " Right
Neutral Surface
Stress Curve Contour
Stress Curve
Stress Curve Contour
Outer Line
Stress Contour
Outer Fibre
Contours of equal Change of length and Stress Intensity
Cylinder
Cylinder
Neutral Surface
Neutral Surface
Zone of Ten.
Zone of Comp
C.Z.
a
b
c
d
e
d'
c'
b'
a'

Fig. T 1

For suppose the rubber shaft be twisted thru such an angle that on the surface points a unit apart on an element have slid by each other half a unit. Then half way between the surface and the center, points on the same radius have slid by each other half a unit—the spiral displacement therefore has half as great pitch on a central cylinder of half the diameter. The difference in pitch shows a twist about each radial line and were it not for the presence of twisting shears about the radius as an axis the torsional shears would set up on the surface principle tensions and compressions at forty-five degrees to the axis of the shaft.

The twist about the axis is balanced by tensions and compressions along the diagonal of the elementary squares which serves to shift the torsional shears along the shaft, and thereby equilibrate the torsional couple at one end by the resisting or power couple at the other end.

Considering the squares deformed into rhomboids on the surface, it is evident that the tension (elongated) diagonal makes an angle greater than 45° to the axis and the compression (shortened) diagonal makes an angle less than 45°.

Where this relation holds the length of the fibers parallel to the axis are shortened since the intensity of the principal diagonal tension equals the intensity of the diagonal compression (as shown in Art. 4, Chapter II) and the compression component would exceed the tensile component parallel to the axis where the angle of the former is less than 45° to the axis.

At $r = .7071\ R$ both diagonals make the same angle, i.e., 45° to the axis, the elements of the cylinder parallel to the axis having a radius .7071 of the half diameter of the prism, remain unchanged in length and the strain upon the cylinder of this radius is a two dimensional strain only.

As r becomes less than .7071 R, the angle of inclination changes sign from the 45° direction and all elements after strain parallel to the axis and at a distance therefrom less than .7071 R are elongated.

The value of .7071 R for the radius of the neutral cylinder is fixed by the static condition of equilibrium that the twist of the principal stresses from 45° about a radius in one direction outside of the neutral cylinder must equal or balance the twist in the other direction inside of the neutral cylinder that the departure from 45° direction may be the least that the elastic restraint permits.

These diagonal tensions and compressions upon all parallel annular rings are principal stresses on these surfaces which shift the shear along the axis of torsion.

2. Direction of the Curves of Greatest Shear and Principal Stress on the transverse sections may be located by the intersections of equally spaced radial and circular coordinates. On any right cross-section of the circular shaft under torsion, draw vertical and horizontal diameters; divide the vertical radius into four equal divisions and draw construction circles there thru dividing the diameter into eight equal parts. Since the applied forces of the torsion couples are 180 degrees apart divide the half perimeter into four equal parts just as the radius was divided into four equal parts and draw diametral lines there thru. Construct right and left spirals thru the intersection of the circles and radials as shown in the right hand corner of Fig. T-1. The curves of greatest shear become tangent to a radial line at the center 180° from their origin on the exterior boundary. The longitudinal shear is zero along all radial lines. Hence all radials are lines of principal stress (compression because of the squeeze of twisting). Likewise, all concentric circles are lines of principal tension.

Circles drawn thru the intersection of the lines of greatest shear between the equally spaced construction lines will thus represent equal contours of bulge but not of volume of longitudinal stress intensity because change in form is a function not only of stress volume but of resisting section as well.

The relation of the coordinates of the shear curves to the contours of bulge or end curvature may be stated as follows:

Equal increments of lever (i.e., divisions of the radius) and equal relative increments of rotation (i.e., divisions of πR) determines the + and − shear curves whose intersections mark contours of equal increments of bulge. The following relations may be utilized to determine the contours of relative stress intensity.

Because the rotational strain about the axis is constant thruout 360 degrees of the area of any transverse section the unit shear stress is also constant hence the stress volume (area times unit stress) increases outward from the center as the area of a triangular sector. Divide this area into four equal parts. The radius of the first equal division will be $R/2$, the second will be $.7071\ R$, the third will be $R\sqrt{.75} = .86602\ R$.

For kinetic invariance of volume there must be as many divisions inside as outside the neutral cylinder which is located at $r = .7071\ R$ (R being the external radius of the shaft and r being the radius of the neutral cylinder, e.g., the stress volume of compression equals the stress volume of tension). The material within the radius $r = .7071$ is therefore elongated and outside of this radius it is compressed or shortened.

To determine the direction of the greatest shears on a longitudinal diametral plane project the points of division just noted across the elevation parallel to the axis, draw *af* and *a'f* and from consecutive intersection of the diagonals with the horizontals draw the verticals *bcd*, *b'c'd'*; then will these verticals be contours of equal longitudinal stress volume. This is true since the angle of rotation is constant from the end to the center and the longitudinal stress volume hence increases following a linear law and its division in the longitudinal plane is similar to that in the transverse plane $L/2$ being the unit of length instead of $D/2$ in the transverse plane. The lines of greatest shear will be curves drawn thru these points of intersection. The principal stresses on the diametral plane are parallel or normal to the axis.

Since the squeeze of twisting is from the surface toward the center the compression which causes the end of the rubber prism to bulge increases in intensity from the outside to the center.

Conversely as the tension by spiralling an element parallel to the axis increases as the distance from the axis, circumferential tensions on all transverse sections increase from the center outward. As this increase of unit intensity follows the linear law and the diameter of the sections affected increases also as the radius of the ring the unit intensity of ring tensions increases as the square of the distance from the axis. Conversely the radial compression which represents the reaction of the ring tensions decreases as the square of the distance from the center from the maximum at the center to zero at the surface. The relative intensities of these stresses are derived graphically in Fig. T-1, lower part.

3. Reverse Summation of Tension and Compression is thus a fundamental characteristic of twist or torsion illustrated by the curve of radial compressions increasing from the outer surface inward and of the circumferential or ring tensions increasing from zero at the center to a maximum at the outer surface.

The Law of Summation of Angular Deformation is from the center toward the end since the twist at one end balances the reaction at the other, elastic equlibrium must occur at the center. Hence angular deformations sum from the center toward the end following the linear law.

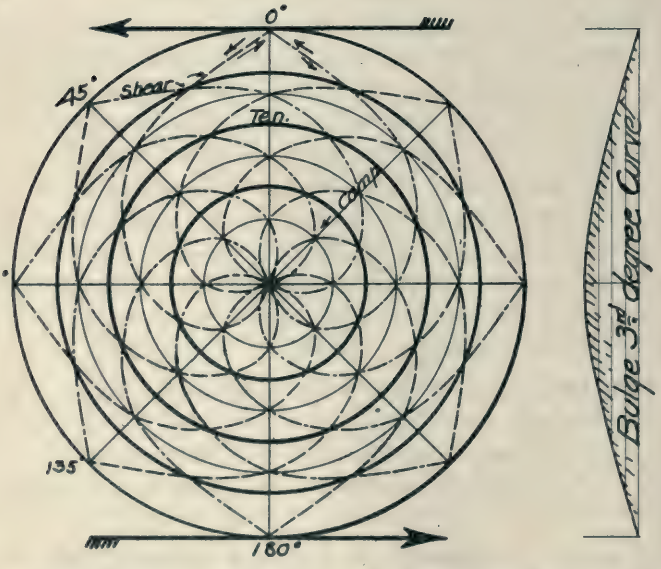
Not only do the radial compressions and circular tensions present the reverse law of summation but the longitudinal compressions and tensions as well.

Thus spiralling of originally straight elements stretching them an increasing amount according to a linear law from the center to the surface tends to produce a conical bulge in transverse sections increasing from the longitudinal center toward the end.

The parabolic variation in intensity of the radial compressions tends to produce a parabolic change in form. Combining these two constituents the resultant radial curvature of the end section would be a third degree curve whose origin is at the axis of the shaft.

While the minute deformations of the simple shaft are of academic rather than practical importance, a clear understanding of the torsion problem is vital to the development of the laws of shear deformation accounting for the great resistance of combined twisting and bending in plates.

FIG. T2. Round Shaft.



4. Comparative Twisting and Bending Resistance of a Round Shaft:

(a) For Torsion (Art. 15, Chapter I, Section I), the maximum intensity of the unit fiber stress *f* equals the maximum intensity of q the unit shearing stress and $q = MR/J$ in which R is the half diameter of the shaft.

(b) Likewise for bending the extreme fiber stress $f = MR/I$.

But since J is numerically twice as great as I a round shaft can resist twice the applied twisting moment that it can support in bending and the same is approximately true for the square shaft.

This relation indicates the importance of careful investigation of the bending stresses in shafting and their practical reduction to a minimum by placing heavy pulleys as near the supporting bearings as possible.

5. Computation of Angular Distortion: Let L equal the length of the shaft subjected to a twisting moment inducing the fiber stress f and let $F =$ the modulus of rigidity; then the angle of torsion ϕ in radians* is

$$\dagger\ \phi = \frac{2fL}{Fd} = \frac{fL}{FR} \quad (L \text{ and } d \text{ being in inches})$$

See Chapter III, Art. 4, Sec. I.

$$\phi \text{ in degrees} = \frac{2fL}{Fd} \times \frac{180°}{\pi} = \frac{1375\,fL'}{Fd}$$

in which for convenience L is given in feet, and d in inches.

6. Diameter "d" of a Round Shaft to Resist a Given Twisting Moment: (M), is computed as follows:

$$\text{Since } M = fJ/R = d^3f/16$$

$$d = \sqrt[3]{\frac{5.1M}{f}}$$

and for a hollow shaft of external diameter d andinternal diameter d_1

$$d = \sqrt[3]{\frac{5.1M}{(1 - d_1^4/d^4)f}}$$

7. The Diameter "d" of a Shaft to Transmit a Certain Number of Horse Power HP at a Given Number of Revolutions N per Minute is determined as follows:

---

[*] A Radian equals $180/\pi = 57°17'44''$.

[†] f the fiber stress intensity equals q the shearing stress intensity in pure torsion. Art. 17, Chap. I, Sec. I.

The work expended in inch pounds per minute is 12 × 33,000 × HP and this equals the statical twisting moment M in inch pounds multiplied by the angular motion $2\pi N$ of the shaft in the same time.

$$\text{Hence } M = \frac{12 \times 33{,}000 \times HP}{2\pi N} = 63024 \frac{HP}{N}$$

Substituting this value of M in terms of horsepower and number of revolutions in the formulas

$$d = \sqrt[3]{\frac{5.1\ M}{f}} = \sqrt[3]{\frac{63024HP}{Nf} \times 5.1}$$

$$\therefore \text{For } f = 5000 \qquad d = 4\sqrt[3]{\frac{HP}{N}}$$

customarily used in shop practice for transmission shafts for mild steel carrying no pulleys.

In the uniform transmission of a given horse power the torsion on the shaft is inversely proportional to the rapidity of rotation.

But such ideal uniformity of torque is not the condition in practical transmission of power and is one of the causes leading to the adoption of such low working stresses as 5,000 pounds per square inch.

Key ways for the attachment of pulleys reduce the effective diameter and strength and this reduction is to be considered in the fiber stress adopted.

Again the shafting is subjected to the bending stress of its own weight between bearings, in addition to the torque of power transmission.

In the empirical or practical formula $d = 4\ \sqrt[3]{HP/N}$ we may regard the customary working stress of 12,000 lbs. per square inch for moving loads reduced a quarter to allow for the weakening effect of the Key way to 9,000 pounds per square inch and this value reduced thirty per cent to cover excess torque over and above the mean and again reduce a fifth to provide for the bending due to the dead weight of the shafts between bearings.

For crank shafts in which the twisting moment at maximum is twice the mean as in the steam engine $d = 5\ \sqrt[3]{5HP/N}$ or an equivalent value of $f = 2{,}600$ lbs. on the basis of uniform twisting moment applied to produce the given horse power.

For mild steel shafting carrying pulleys as in the textile mill with bearings eight to ten feet centers the practical rule for d is

$$d = 4.5 \sqrt[3]{HP/N}$$

8. The Resultant Principal Stress limiting or determining the working capacity of the shaft in transmitting power is to be found by combining the state of stress due to the bending moment of the dead weight of the shaft and any pulleys between bearings with that caused by the torque or twist incident to transmission of the power.

Consider the point 0 (top or bottom) of maximum positive bending of the shaft. Since it lies in the vertical plane thru the axis of the shaft, the lateral displacement is zero.

The distortion of the outer fiber from pure longitudinal compression and transverse torsion only are to be considered and these may be treated as follows:

Let P_x equal the resultant principal compression and P_y the resultant principal tension, f being the extreme fiber stress from bending and q the greatest unit shear stress from pure torsion. Then in the resultant ellipse of stress from the principles of Art. 20, Chapter I, Sec. I.

$$(1) \quad \frac{P_x + P_y}{2} = f/2 =$$

the radius of the component state of fluid stress. And

$$(2) \quad \frac{P_x - P_y}{2} = \sqrt{q^2 + (f/2)^2} =$$

the radius of the component state of right shearing stress and the resultant maximum unit shear as well.

The component states of right shearing stress combined in $\dfrac{P_x - P_y}{2}$ are at 45° to each other.

From (1) and (2) we may write

$$P_x = \sqrt{(f/2)^2 + q^2} + f/2 \quad \text{compression} \quad \ldots\ldots \quad (3)$$

$$-P_y = \sqrt{(f/2)^2 + q^2} - f/2 \quad \text{tension} \quad \ldots\ldots \quad (4)$$

Now if i be the angle which the direction of P_x makes with the axis of the shaft, then

$$\text{Tan } 2\,i = 2q/f, \text{ or in another form} \ldots\ldots \quad (5)$$

$$\text{Tan } i = \sqrt{1 + f^2/4q^2} - f/2q = -P_y/q \ldots\ldots \quad (6)$$

Equation (5) may be proved readily by applying the parallelogram principle to the bending and twisting moments which produce f and q respectively. Sec. I, Chapter I, Art. 6. Were the resistance to bending equal to that of torsion then tan i would equal T/M in which T is the applied twisting moment and M the applied bending moment, but as T is only half as effective (see Art. 4) as M in producing stress intensity Tan i equals half the angle whose tangent is $T/M = 2q/f$.

The point antipodal to 0, top or bottom as the case may be in a continuous shaft is the point of maximum negative bending and the same formulas with reversed sign apply with practical accuracy, since the ratio of L the distance between bearings divided by the diameter of the shaft d is so great that the reduction of the tensile bending stress by the interference of detrusion discussed in Sec. II may be disregarded as negligible.

The longitudinal stresses of pure torsion parallel to the axis because they equilibrate each other on either side of any diametral section produce no bending couple; hence do not increase the distortion of pure bending and may therefore be disregarded.

Verification: Equations (3) and (4) may be readily verified by considering the values of f and q at the limit. Thus if $f = 0$, $P_x = q$, while if $q = 0$, $P_x = f$, as it should be.

The bending stress in the plane of the surface may be obviously separated into component states of fluid stress and right shearing stress of half the intensity of the bending stress, oriented so that the principal stresses of like kind coincide. Then $P_x = f$ and $P_y = 0$. To combine the component states with the state of right shearing stress due to torsion it may be noted that there is no shear in the state of fluid stress and we have only to combine two states of right shearing stress to determine the greatest shear stress $= P_x - P_y/2$.

The general formula for combining two states of right shearing stress in which P_1 is the principal stress of the first state and P_2 is the principal stress of the second state of shearing stress of like kind, then $P_x^2 = P_1^2 + P_2^2 + 2P_1 P_2 \cos 2x_1 ox_2'$ in which ox_1 represents the direction and intensity of P_1 and ox_2 the direction and intensity of P_2, and the angle between them x_1ox_2. Now when $x_1ox_2 = 45$ degrees $\cos 2x_1ox_2 = 0$ and $P_x^2 = P_1^2 + P_2^2$ which corresponds to equation (2).

It will be a good exercise for the student to prove this equation graphically by the parallelogram principle and if any difficulty is found in so doing reference may be made to problems 19 and 20, page 116 of Researches of Graphical Statics of H. T. Eddy, (1878).

The problem of determining the tangent of inclination of the resultant principal stress to the axis of the shaft is similar to that of determining the inclination of principal stress to the neutral plane of a beam, as noted by Rankine, Arts. 310 and 311 of his Applied Mechanics.

To clarify the matter of relative intensity of the working stresses it may be desirable to figure the relative intensity of the resultant maximum stress for a wide variation of the values of q and f. If $f = \frac{1}{2} q$ then the resultant maximum principal stress $= 1.28 \times q = 2.56 \times f$. If $f = 2 \times q$, then the resultant principal stress $= 2.1414 \times q$ and $1.207 f$. If $f = 3 \times q$ then the resulting principal stress $3.3 q = 1.1 f$. If $f = q$, $P_x = 1.618 f = 1.618 q$.

9. Limitations of Practical Formulas: From the foregoing values it at once appears that the formula $d = 4.5 \sqrt[3]{HP/N}$ for the diameter of shafts carrying pulleys provides for a bending moment no more than half as great as the torsional moment or for a fiber stress in bending equal to that caused by the torque or twist.

Accordingly, the line shaft carrying pulleys proportioned by this practical or empirical rule should be investigated to determine that the heavier pulleys or those producing relatively the greatest bending do not produce moments largely exceeding this limitation.

Kent Mechanical Engineers Pocket Book, p. 869, gives the rule for $d = \sqrt[3]{50\, HP/N}$ for continuous line shafts of mild steel without pulleys, added allowance being made for the weakening effect of Key seats, which is almost identical with the rule before given allowing 1/11th increase in diameter for the weakening effect of the Key way.

Torsional stress for the uniform transmission of a given horse power is inversely proportional to the rapidity of rotation whereas the bending stress caused by the weight of shaft and pulley does not decrease in like manner.

Therefore, simple practical rules are approximately correct only for the limited range of velocity common in practice, such for example as 180 R.P.M. for machine shop practice to 250 to 300 for the line shafts of wood working and textile mills, etc.

If for example we figure the diameter of one shaft to transmit the same horse power 1,000 R.P.M. and another at 125 R.P.M.

the diameter of the latter will be twice that of the former under the practical formula but the bending moment from the weight of the larger shaft is four times as great as for the smaller, but on the other hand its moment of inertia is eight times as great, hence the fiber stress from bending is but half as great for the larger shaft, and the practical rule applying to the slowly moving shaft cannot apply to the eight-fold increase in speed. But if the range of variation is between 180 to 300 R.P.M. the corresponding variation in the resulting principal stress will be limited to such a moderate amount as to be negligible in comparison to the probable accuracy in estimating the variation in torque above the mean in the practical problem.

10. Simplified Computation: As the distance between bearings or unsupported length of the shaft, and weight of pulleys varies the bending component of the limiting stress varies while the torsional component does not.

This variable relation may be investigated as follows. Let K equal the ratio of the bending moment divided by the twisting moment so that $K = M \div T = f/2q$.

Let d'' = the diameter of the shaft calculated for the twisting moment when $M = 0$ and $\mathbf{n}d' = d$ equals the diameter required to resist the combined bending and twisting.

Then from the relative values of the maximum combined stress Px computed for the relative varying ratios of f the bending stress component divided by q the torsional stress component we may tabulate the range of values of K and **n** as follows:

When $K =$ 0.25	0.50	.75	1.50	2.00	3.00
Then **n** = 1.17	1.26	1.34	1.49	1.62	1.83

Since the twisting moment computed from HP/N represents a mean and not the maximum value to be provided for the excess of the maximum above the mean is to be estimated or approximated from the character of the service and the component stress q computed in accord therewith *before* combining it with the bending stress to determine the resultant maximum.

The divergence of the maximum torque from its mean value as been found for engine shafts approximately as follows:

Single Engine Maximum Torque 2 times the mean
Compound Engine Cylinders at 90°, Max. Torque 1.4 × mean
Double Engine Cylinders at 120°, Max. Torque 1.8 × mean

In the long transmission shaft of the textile mill, the starting torque exceeds the mean by from thirty to thirty-five per cent, while in the ordinary machine shop it may be double this amount under possible combinations of the maximum number of machines operating under the heaviest service.

11. Regulation of Torque Variation: Very wide variations in torque are eliminated in good practice by placing a fly wheel of suitable weight on the far side of the take off pulley from the power end of the shaft. With such an arrangement, the energy stored in the fly wheel at peak loading may momentarily increase the power torque on the take off pulley to double its mean value without increase of the fiber stress on the shaft which at the take off pulley presents a state of double shear in contradistinction to the single shear of uniform resistance.

12. The Limiting Deflection of shafting for smooth running may not exceed about 1/100 of an inch for each foot of span between bearings. Where this ratio of deflection may be exceeded under the weight of the shaft and attached pulleys, it is necessary to increase the shaft diameter to secure the needed stiffness or reduce the span between bearings.

Now deflection increases as a coefficient times the cube of the length, times the weight, divided by the moment of inertia or relative stress resistance, hence L the limiting span is:

$$L = B\sqrt[3]{d^2}$$

in which the coefficient B equals 65 for bare shafting and 50 for shafting carrying ordinary pulleys.

13. Summary: The reason underlying practical rules for proportioning of shafting and the limitations which govern their application have been developed, and the theory of location of the fly wheel with respect to economic design of shafting for transmitting variable torque developed in such wise that the problem of shaft design is removed from rule of thumb practice to rational procedure.

From the elastician's view point, the preceding discussion shows that in a circular shaft under torsion, there is one transverse plane which suffers no longitudinal deformation where the power couple and take off couples balance. All other transverse planes suffer bulging distortion to the end of the shaft. This bulging distortion is symmetrical, i. e., every point having the same radial distance from the axis lies in the same transverse plane normal to the axis, i. e., the transverse distortion is symmetrical in the cylinder under torsion but as will be seen later, such symmetry is lacking in the square or oval prism under torsion.

CHAPTER XIII

TORSIONAL ANALYSIS OF SQUARE, RECTANGULAR, TRIANGULAR AND OVAL PRISMS

1. That action and reaction may equal each other, applied twisting moments about the axis of any prism regardless of the form of its cross section must be met by uniform resistance through 360 degrees about that axis. In the case of a square or rectangular prism, the radii along which this constant total rotational force of resistance is distributed, differ in length, therefore the rotational forces in planes parallel to the axis and normal to the respective radii compared differ in intensity. All transverse sections normal to the axis will thus be distorted longitudinally throughout the entire length of the prism. Since the direction of rotation of these couples is the same, alternate longitudinal bulges and depressions will appear in alternate octants of all originally normal cross sections of a square shaft under torsion, since the unit intensity of the resistance alternately increases and decreases in passing circularly through octant to octant.

This may be shown experimentally as follows. Upon the surface of an inch and a half square prism of soft vulcanized rubber, six inches long draw lines a quarter of an inch apart parallel with and normal to the edge. Grasp the lower edge of the prism firmly and twist in a clockwise direction (as viewed from above). Squares along the center of the sides by the twist have become deformed into rhomboids, the rotational distortion about the horizontal axis being clockwise, as viewed from the side, when the direction of the twist is clockwise, viewed from above. The normal lines still intersect the edge at right angles and the squares at the edge unlike those in the middle are but little deformed.

The experiment shows that **In the Plane of the Sides the Greatest Shear Distortion is Nearest the Axis.**

The greatest shear distortion along a given radius is found in a plane normal to the radius as appears from the preceding discussion of round shafts. Examining a point midway between the center and edge as at the extremity of a radial line at 15 degrees to the short diameter, it appears that the rhombic distortion of the squares is but little less than that at the middle of the side. Hence, the shear distortion normal to the 15 degree radius will be greater than at the surface at the end of the short diameter.

2. The Curves of Greatest Shear are determined in the same manner as those of the round shaft, i. e., equal increments of lever circularly or tangentially and equal relative divisions of the angular change in direction of the tangential force provide the coordinates for the construction of the curves.

In a round shaft, the tangential forces change direction 180 degrees in 180 degrees of circumference, four divisions of the radius and four of the semicircumference provide the coordinates.

In a square shaft, the tangential forces change direction 180 degrees in a 90 degree angle, hence the radial coordinates are 22 degrees 30 minutes apart, instead of 45 degrees apart as in the round shaft. In other words, the shear spirals of a square shaft have twice as sharp a pitch as in a round shaft.

In a rectangular shaft, the radial divisions will be one fourth the angle between the diagonals, for example, if opposite the long face this angle be 120 degrees, the radials will be 30 degrees apart and opposite the short face 15 degrees apart.

Obviously with such a difference in the radial coordinates, the compartments of alternate bulge and depression will undergo a change from octantal to quadrantal divisions. This change occurring when the long side is 1.454 times the short side as determined by St. Venant.

In a shaft whose section is an equilateral triangle, the tangential force changes direction 240 degrees in the 120 degree radial angle subtended by the side, hence for four divisions of the short radius the circumferential angle of the radial division is 120 degrees divided by four or 30 degrees.

For the oval, while the tangential forces as for the round section change 180 degrees in 180 degrees of circumference, the RATE OF CHANGE is not a constant, hence the angularity of the radial divisions is to be varied so that the length of each circumferential division subtends an equal angle of change in direction of the oval circumference.

Alternate compartments or areas of bulge and depression will be quadrantal like the elongated rectangle, but the curves of the greatest shear will differ from the rectangle in that they will have but half as great pitch, i. e., a difference similar to that between the square and the round section.

Figures *Ts*, *Tr*, *Tt*, and *To*, present typical shear curves for the four respective types of sections.

Fig Tr

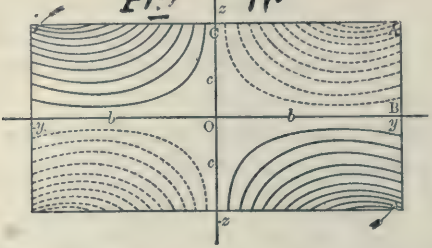
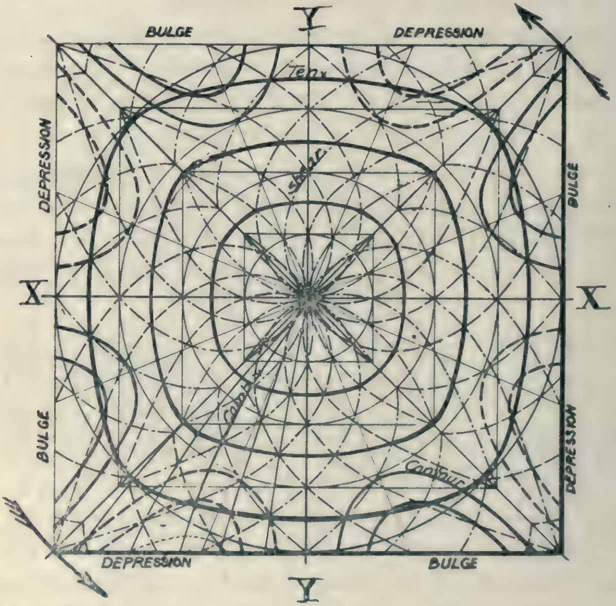


FIG TS. Square Shaft.



Fiq To.

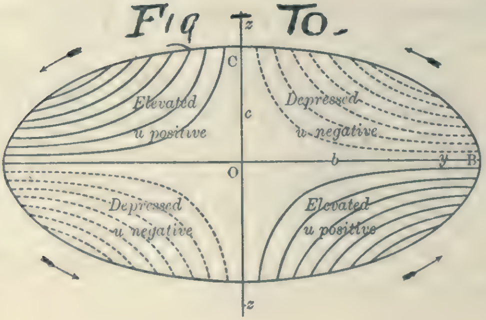
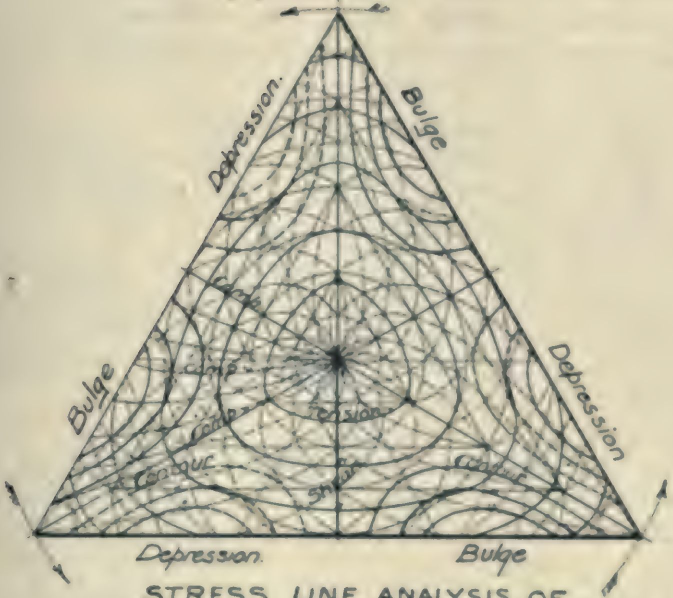
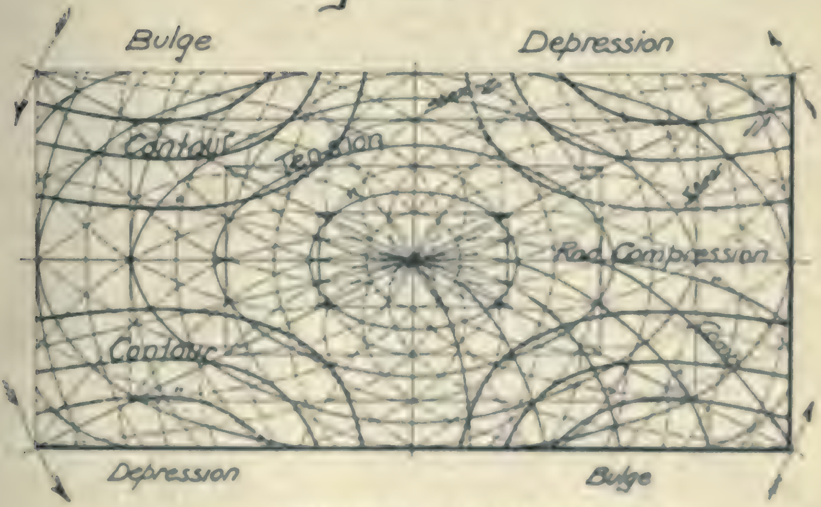


Fig - Tt.



STRESS LINE ANALYSIS OF
Triangular Shaft under Torsion.

Fig. Tr.



STRESS ANALYSIS OF
Rectangular Shaft under Torsion

It is to be noted that an abrupt angular change in the tangential boundary gives rise to a change in sign of the shears and a cusp in their formation so that some of the coordinated shear spirals stop short of the surface upon the radial plane through the angular edge and the axis of the shaft.

3. Curves of Principal Stress as shown in Art. 18, Chapter 1, Sec. 1, bisect the angle between the curves of greatest shear.

In the round shaft on any transverse section the principal stress curves are radial and circumferential.

For the square prism, the diagonals and short diameters are straight line radial principal stress curves, while other radial principal stresses follow curved lines. The circumferential principal stress curves may be termed four point ovals, i. e., with elongated diameters at 90 degrees apart.

For the triangular prism, radials to the corners and normal to the sides only are straight line radial principal stress curves, while the circumferential principal stress curves normal thereto, are oval with elongated diameters opposite the apices of the triangular section.

4. The Relative Rigidity of the Square and Round Shaft is proportional to J, the polar moment of inertia of the section under the ordinary assumption that shear strain is proportional to the distance of the fibre from the axis of the shaft, i. e.,

The elastic strength will vary as J/C, i. e.,

For the round $T = .1963\ d^3S$
For the round $T = .2357\ d^3S$
For a round shaft $\phi = 10.186\ TL/Fd^4$
For a square shaft $\phi = 7.22\ TL/Fd^4$

in which ϕ is the angle of twist in radians T, the torque, L the length and d the diameter of the round or side of the square in inches and F the modulus of rigidity or shearing modulus.

The stress in a square or rectangular shaft differs from that assumed as the basis of the preceding computation as shown by experiment. Thus square or rectangular bars that have been marked off with series of equally spaced longitudinal and transverse lines after twisting show a distortion of the squares into rhombuses, the greatest distortion on a face occurring at the middle of the sides, while the transverse lines, as nearly as can be observed, remain perpendicular to the edges of the bar. Since shearing stresses in the plane of the face are proportional to the distortion it appears that the shearing stress on the surface is greatest at the middle of a side of the square bar; and greater at the middle of the wide side of a rectangular bar than at the middle of the narrow side, and that there is no shearing stress at the corner.

The intensity of the longitudinal shear which causes the bulge and depression of alternate octants of the square prism under torsion may be considered to increase as the ordinates of a parabola from the edge to the center and as the resisting section of the octant about the short diameter increases as the ordinates of a triangle, i. e., a linear law the deformation or strain will vary as the ordinates of a third degree curve.

The total moment of the longitudinal displacement about the short diameter would therefore be $.3478 \times 2 \times 5/8\ T = .434\ T$ and the relative rigidity of the round and square section would be as 1: 1.434.

or $\phi = 1.0186\ TL/Fd^4$ for a round and

$$\phi = \frac{1.0186^2}{1.434}\ TL/Fd^4 = 7.1 \times TL/Fd^4 \text{ for the square section}$$

The angular distortion is greatest in a plane normal to a radius, hence, as the distortion in the plane of the face of the square shaft decreases at first but slowly from the center toward the edge, the greatest distortion will occur normal to some radius between the short diameter and the diagonal which is found at an angle somewhat less than 20 degrees, and the corresponding elastic strength would be 1.06 times that of the round shaft as found by St. Venant.

5. Strength of the Rectangular Prism Under Torsion:

St. Venant gives the following approximate formula for the rectangular prism under torsion:

$$T = b^2h^2q_1/(1.8b + 3h)$$

In which h and b are the long and short sides of the rectangle, and q_1 the greatest shearing stress.

Merriman gives the formula

$$T = \frac{2}{9} b^2hq_1$$

which for values of h greater than $2b$, gives smaller values than are found by St. Venant's more exact method.

Woods in his Strength and Elasticity of Structural Members replaces 2/9 of Merriman's formula by a variable coefficient (k) varying with the ratio of h to b as follows:

*h/b	k	h/b	k
1	0.208	3.5	0.275
1.5	0.231	4	0.282
2	0.246	5	0.292
2.5	0.258	10	0.312
3	0.267		

Recapitulation and Remarks: The warping of the normal section of a prism under torsion is analyzed by Love, Arts. 221, 223, of his Mathematical Theory of Elastic Solids, while a digest of the original investigation of St. Venant is given in English in Vol. II, part I. History of Elasticity and Strength of Material by Todhunter and Pearson, pages 1-51. (On page 59 appears St. Venant's important investigation of the ordinary theory of flexure demonstrating that for the important practical cases of continuous loading, the Benouilli Eulerian hypotheses is not even a close approximation to the truth. This question has been treated in our Sec. II in a simple manner and the deviation from exact values investigated and determined.)

The graphic analysis presented in the preceding chapter involves only simple conceptions of proportion in addition to the elementary principles governing states of stress in a plane or solid investigated in Sec. I, Chap. I.

Since displacement is the joint product of shear and principal stress from the known direction of the curves of greatest shear and principal stress, St. Venant's contours of bulge and depression are readily drawn graphically by bisecting the angle between these curves while eliminating tedious and complicated mathematical work.

While it is true that the state of stress in a plane at a point is fully defined by two orthogonal principal stresses, such definition conveys no precise conception of the nature and amount of displacement resulting from the given state of stress.

Thus rupture (the result of exceeding a limiting displacement), is a joint product of shear and principal stress and the plane which it will follow is known when the direction and sign of the curves of greatest shear and principal stress are known. The multifarious explanations of rupture offered in works dealing with the resistance of materials is the outgrowth of the inherent haziness of this abstract definition.

---

* Bulletin No. 3, Section No. 9, University of Toronto, by Messrs. Young, Sagar and Hughes, presents experimental data on plain and reinforced rectangular concrete prisms under torsion.

The concise grasp which graphic presentation of states of stress achieves should enable the student to appreciate and profit by the numberless examples of perfect economic distribution of material in nature, to be found not only in the vegetable kingdom but in the complex structure of the bones of the higher animals which, from a mechanical standpoint, are a system of levers requiring a high degree of strength and relative lightness as well. The excellent presentation of this matter by W. J. Ibbetson in an appendix of his treatise on the mathematical theory of elasticity should prove of interest to the student.

CHAPTER XIV

ANALYSIS OF COMBINED BENDING AND TWISTING RESISTANCE OF HOMOGENEOUS FLAT PLATES

1. The Characteristics of Plate Action will be first developed from test of celluloid plates under load and later the phenomena will be analyzed by the elastic theory.

(a) Support a 12″ x 12″ x 1/16″ thick plate upon opposite edges first drilling a 1/16th-inch hole thru the center of the plate. Pass a piece of twine thru the hole knotting the end. Suspend a two pound weight on the twine and measure the central deflection and note that the deflection at the edge at mid span differs inappreciably from that at the center at mid span. Reverse the plate and record the mean of the deflections.

(b) Support the plate on four sides instead of two and apply the same load suspended at the center; record the deflection. Turn the plate bottom side up; load again as before and take the mean of the readings.

Comparing the two deflections, that of experiment (a) is slightly more than four times as great as that of experiment (b). The applied bending moment in (a) is $WL/4$. The applied apparent moment in (b) is $WL/8$ in each direction normal to the sides of the plate, less the negative twisting moment.

Reduction of the apparent moment by fifty per cent should render the deflection half as great in (b) as in (a) if deflection and apparent moment could be compared on a proportional basis in the two respective cases.

In experiment (a) the curvature was one way or cylindrical but in (b) the curvature was two ways and the plate had been warped or twisted into dish shaped curvature again halving the deflection. From this result we may infer that **warping or twisting plays an equal part with two way bending in producing the surprising strength and stiffness of plate action in dish shaped curvature.**

Since a plate may also be twisted into saddle shaped or reverse curvature the question at once arises whether the warping or twisting is equally efficient in saddle as in dish shaped bending.

Drill the square celluloid plate near each corner locating the hole three sixteenths from each side. Suspend the plate level by vertical strings thru the diagonally opposite holes (knotting the twine on the under side of the plate).

(c) Suspend a two pound weight from a string passed thru the center of the plate and note the deflection. Reverse the plate load and record the deflection taking the mean.

(d) Remove the load from the center and suspend one pound from each corner and measure the central deflection. *Reverse, repeat and record the mean.

Compare the mean deflections of experiment (c) and (d) and the deflection of the center of the plate is found one half as great in (d) as in (c) and that of the corners of the plate just as great because by symmetry the bending is equal and opposite.

Accordingly, **the twist in equal but opposite bending opposes a resistance to deflection similar to the twist of equal and like bending.**

Equal bending at 90 degrees is called SYNCLASTIC or SPHERICAL curvature when the bending is of like sign and ANTICLASTIC when the equal bending at right angles is of opposite sign.

2. Separation of Bending and Twisting in Plate Action: Rotational shear strain in bending is uni-axial or about parallel axes in bending resistance but torsional shear strain is bi-axial as investigation of the round and square shaft disclosed in Chapter XII. Accordingly, in the quadrant of any plate bent either in dish shaped or anticlastic curvature if the principal bending is about the horizontal axis of XX and the total effective bending moment is M_x, then will the total applied torque about YY equal numerically half M_x, and the total applied twist about ZZ will also equal half M_x, therein explaining the equality of bending and twisting effects found in experiments (b) and (d).

These applied twisting and bending moments are in general opposed by distributed resistances thru the entire area of the plate but by greatly increasing the torsional rigidity of an element about the vertical or ZZ axis in anticlastic curvature a separation of the area into zones separately resisting bending and twist will result.

Take a very thin square celluloid plate of about the order of one five hundredth of thickness to span length; suspend it by vertical silk threads at diagonally opposite corners and apply equal but not too heavy loads at the other corners. *Half the area will remain unbent and half will be bent. Thus if Fig. *P*1 represents the plate supported at S and S' by vertical silk threads and loaded at P and P' by equal, very light loads suspended by silk threads, then will the square *abcd* unbent represent the zone of twisting resistance and the triangles *adS*, *cbS*, *cdP*, and *abP'* represent the zones of pure bending.

---

* This precaution eliminates the error or effect of residual strain in the plate.

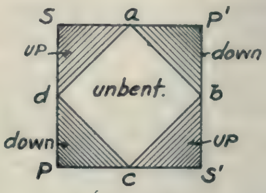


Fig P1.

This experiment indicates not only that twist and bending play equal parts in plate action but also that the twist stores energy in a manner not productive of deflection. Moreover, it demonstrates conclusively that the twist in the horizontal plane of vertical elements about the vertical axis ZZ must be reckoned with in any rational analysis of plate action.

The equations of La Grange upon which much of the complicated mathematics of plate resistance heretofore published is based, fail to do this and hence results deduced therefrom are incorrect.

3. Circumferential and Radial Bending Resistances combined with uniform circular compression over the full depth of the plate increasing from zero at the center to the periphery may be experimentally demonstrated as the result of twisting and bending in a uniform circular plate supported at its periphery and centrally loaded as follows:

Compare the stiffness of a circular plate one sixteenth inch thick and nine inches in diameter under a given central load with a duplicate plate slit with a fine saw from the periphery on radial lines forty five degrees apart to within one and a half inches of the center. Radial fibers are all intact but the circumferential resistance is eliminated by the saw kerfs so that the deflection is about twelve times as great after cutting as before.

By increasing the load the slits close up at the circumference and after they abut the plate is as stiff as the one uncut.

---

[*] This experiment of Art. 2 appears to have been first discussed in Natural Philosophy, Thomson and Tait, 1867. The threads must be vertical and loads not too heavy or unstable equilibrium results.

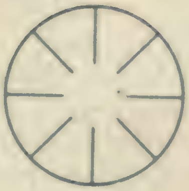


Fig P2.

The fact that the kerfs snap together at the outer edge of the plate and the resistance to further deflection is then increased many fold by the compression of the abutting segments conveys a definite conception of the nature of synclastic resistance. It shows that an important element of such bending resistance is the horizontal circular compression of the plate as a whole about the outer edge superposed upon the circumferential bending resistance. Conversely, in anticlastic bending stretching takes the place of this compression in synclastic curvature.

Examining the radial sectors between the slits of the plate *P*-2 as bent it will be noted that these radial sectors remain unbent circumferentially and as the corners come together the bearing of the plate on its ring support is only at and near the slits. This phenomena indicates THREE ELEMENTS OF RESISTANCE IN PLATE ACTION:

(1) **Radial bending resistance.**

(2) The resistance of circumferential bending which causes the plate to bear upon its support along its periphery.

(3) Circumferential compression produced by twisting of vertical elements squeezing the slit sectors together circumferentially. This compression being greatest at the periphery and reducing toward the center of the plate.

4. Phenomena of Buckling of Thin Plates: The third element of resistance, circumferential compression, is responsible for the phenomena of buckling in the thin plate which then acts after the manner of a long column. For example, in the thin circular plate at its periphery horizontal shears in vertical planes circumferentially pass thru zero at the mid depth of the plate but the fact that circumferential fiber compression stress area is greater than the tensile area causes the plate if thin, to buckle or assume a scalloped curvature under this excess fiber compression. The downward displacement of the neutral plane as in beam action toward the support measures this excess circumferential compression over the tension, the linear law of fiber stress intensity being unchanged by twist.

A rectangular sheet of roofing tin supported on the four sides deflects but little before it buckles with a snap in conical form normal to the middle of each side. The square celluloid plate 12 x 12 x 1/16th-inch thick if overloaded at the center springs up from its support at the middle of each side and as soon as it buckles the deflection increases to a very marked degree because the circumferential compression is relieved by this buckling and an important element of plate resistance is thereby eliminated.

In the circular plate one buckle only occurs against four in the square plate supported at its periphery and its position is here unstable, capable of being rotated around thru 360 degrees. The fact that whether the plate is rectangular, square or circular, when thin, buckles in conical form indicates that the travel of the load is along radial lines whereas the principal tensions and compressions which are caused by the twisting shears, i.e., horizontal shears in vertical planes and the bending shears, (vertical shears in vertical planes), follow another and a different law.

5. Stress Curves and Displacement: Principal stresses, like contours, follow curved lines, changing directions from point to point, not abruptly but by small increments. Eddy, in a chapter devoted to the theory of internal stress in his Graphical Statics of 1878, regards it as a physical axiom that

> "The principal stresses upon an ideal plane of division which traverses any given point of a body cannot change suddenly either as to direction or magnitude while that plane is gradually turned in any way about a given point. A sudden variation can only take place at a surface where there is a change of material." (See analysis of reinforced concrete beam, Sec. 11).

A different law, however, governs the curves of displacement and equi-potential. These curves in vertical planes change abruptly in passing from the compression zone to the tension zone. In like manner, the curves of displacement change abruptly on the surface of the plate where the direction of the twisting shears change sign.

Vertical planes at which twisting shears change sign may be regarded as planes of greatest or least curvature in the plate as a whole. For example, in anticlastic curvature, with equal and opposite bending, the twisting shears change sign in a plane at forty five degrees to the planes of greatest or principal curvature of the plate. In other words, where the curvature changes from positive to negative in the warped surface the twist is a maximum and curvature passes thru zero. The location of such a plane in the warped surface of the plate is the first step in the analysis of of the curvature of continuous plates. Where the twisting shears change sign but neither curvature passes thru zero is a characteristic of the plane of greatest curvature of the plate. Such a plane in passing thru consecutive points of the plate is the plane of greatest curvature for each of the points and hence for the plate as a whole. In a plate bent in equal reverse curvature the two planes of principal curvature of the plate are at right angles to to each other.

In a plate bent to a surface of revolution every meridian plane is a plane of principal curvature because there is no difference in the curvature between consecutive meridian planes.

In a plate bent in reverse curvature, if the positive and negative curvatures are equal the planes of zero curvature are obviously at 45° to the planes of principal curvature. But if the curvatures on these planes are numerically unequal, then the planes of zero curvature rotate toward the sharper principal curvature.

6. Separation of States of Surface Stress and Curvature of Uniform Bent Plates into Simple Components: As a uniform homogeneous plate may be bent in any conceivable manner the state of stress may likewise vary. If, however, the curvature be spherical the state of stress is a fluid stress on top or bottom surfaces and if the curvature be anticlastic (orthogonally equal and opposite bending) then the state of stress is a right shearing stress.

Accordingly as any state of stress whatever in a plane at a point can be separated into a state of fluid stress and a state of right shearing stress (see Arts. 11 and 16, Chapter 1, Section 1) the stress at any point on the surface of a uniform plate bent elastically in any manner whatever may be separated into component states of fluid stress and right shearing stress of proper intensity. Likewise the curvature may be separated into component states of spherical and anticlastic curvature defined by suitable radii.

7. Separation and Composition of Curvatures: The manner in which these states of curvature combine may be illustrated as follows:

Spherical curvature and anticlastic curvature of equal radii superposed upon each other result in plane cylindrical curvature. Combination of spherical curvature and an anticlastic curvature of smaller radius results in a dish shaped curvature having unequal radii at the planes of principal curvature, while the continuous plate supported on equally spaced columns in square formation may be considered as made up of spherical curvatures and anticlastic curvature at forty five degrees to each other, and in like manner any conceivable curvature may be separated into spherical and anticlastic curvatures superposed upon each other.

8. **The Sum of the Reciprocals of the Radii of Curvature on any Orthogonal Pair of Planes Normal** to the surface at a point of an elastically bent uniform flat plate is a constant, since the surface curvature can be separated into component states of spherical and anticlastic curvature in which the sum of the reciprocals of the radii on orthogonal planes is a constant in the resultant curvature their sum must likewise be a constant.

This proposition is known as Euler's theorem. It is analogous to the principle of conservation of fiber stress areas, Art. 21, Chap. 1, Sec. 1, and to the proposition Art. 20, that the sum of the normal stresses on any two orthogonal planes normal to the plane of stress thru the point considered are equal.

9. **Planes of Principal Curvature at a Point of a Uniform Bent Plate are Orthogonal to Each Other.** Because the state of stress at a point in a plane is fully defined by two principal stresses orthogonal to each other, Art. 17, Chapter 1, Sec. 1, the curvature of the uniform flat plate elastically bent by any state of stress at any point on its surface is fully defined by two principal curvatures which in terms of $1/R$ are algebraically the greatest and the least ratios found by revolving the normal plane thru the point 180 degrees.

10. **Twisting Shears Change Sign in Passing Planes of Principal Curvature:** Since planes of principal curvature of the plate as a whole are planes of greatest and least curvature the curvature becomes less as the normal plane of section is revolved in either direction from the plane of principal curvature; hence the twisting shears giving rise to this opposite change of curvature must be of different sign. This relation in double curvature is analogous to that in cylindrical curvature where the shears pass thru zero and change sign as the bending moment passes thru a maximum value.

Corollary 1. The curvature of a plate in the plane of principal curvature like that of a beam is determined solely by the bending moment separate and distinct from the total applied statical moment which may be compounded of twisting and bending moments.

Corollary 2. The equilibrium of horizontal shears in horizontal planes and horizontal fiber stresses takes place in vertical planes of principal curvature of the plate, just as in beam action treated in Chapters VII and X, Sec. 11. Since twisting shears pass thru zero at these planes.

Consequently, the curves of greatest shear and principal stress in these vertical planes may be drawn in the same manner and by applying the same reasoning as was done in the stress line analysis of beams, Chapter X, Sec. 11.

11. Effective Moment: as the cause of bending may be defined as the algebraic sum of all the moments between the support and a section about parallel axes.

It is related to deflection, thru the double summation of moment areas and slope areas and hence deflections along planes of principal curvature are proportional to the effective moments in different cases when these summations over EI involve similar areas.

Planes of symmetry enable the separate computation of twisting and bending moments and their algebraic sum represents the effective moment by which relative deflections may be figured under the limitation noted above.

(a) *Uniform Load:* Let $ABCD$, Fig. $M1$, be a square plate supported at A and B having a diagonal span L, the supports D and C being assumed at too low an elevation to take any load at all. A and B each support half the load. The center of gravity of the half uniform load, $W/2$ supported by B, is $L/3$ from B, and its moment is $WL/6$.

Suppose now the supports A and B are lowered until the reactions at ABC and D are equal; then each carries $W/4$, Fig. $M2$.

Interference of Moments at right angles now occurs accompanied by warping or twisting of plate. Again, in the division of the load area between the four supports the form of the area supported by each post has changed from triangular, Fig. $M1$, to rectangular, Fig. $M2$. In $M1$ the load area is half the plate with its center of gravity $L/3$ from the support. In $M2$ the load area carried by each of the four supports is a quarter of the panel as $gOhb$ to B with its center of gravity or lever for moment $L4$ from B. Therefore the positive applied moment of $WL/6$ in $M1$ has been reduced to $WL/16$ in $M2$. It has first been halved in keeping with four supports instead of two and been further reduced by a fourth by the relative change in position of the center of the load area from the support to $\frac{3}{4}$ of half $WL/6$ or $\frac{3}{8}$ of $WL/.6$.

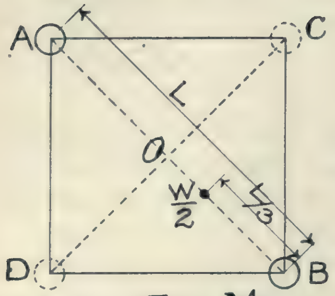


FIG M1.

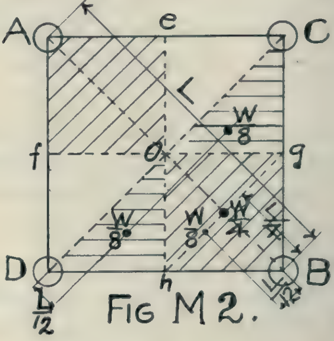


FIG M2.

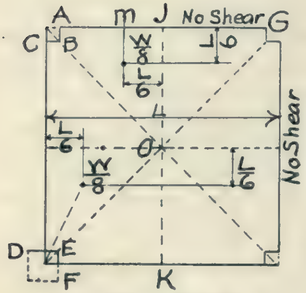


FIG M3.

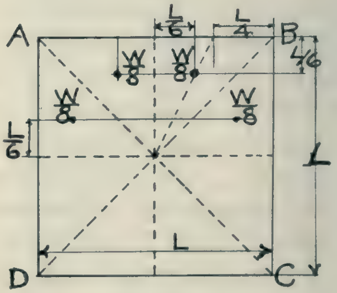


FIG M 4.

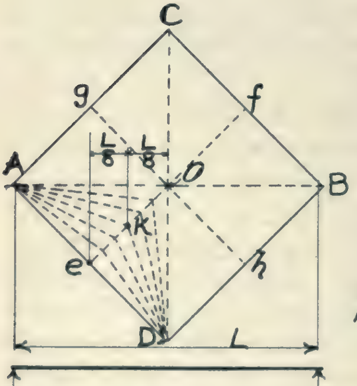


FIG M.5

Zero Shear Lines.

(a) Uniform Load. Supports A, B, C, D.

(b) Equal. Loads C. and D " A. and B.

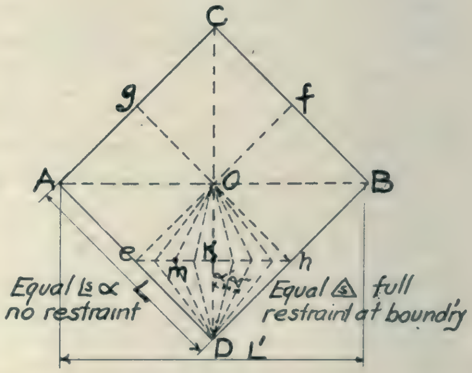


FIG M 6

Zero. Shears Corner Supports:

Concentraled Load at O.

This $\frac{3}{8}$ of $WL/.6$ is further reduced by the negative twisting moment about CD, which is $2 \times W/8 \times L/12 = -WL/48$ as appears from the condition that by symmetry fg and eh are zero vertical shear lines hence triangles CgO and DhO with centers of gravity $L/12$ from CD have a negative twisting moment of $WL/48$ just figured; thereby reducing the effective moment of the same uniform load when carried by only two supports A and B by three quarters when the four supports carry equal shares of the load.

(b) *Concentrated Load Anticlastic Bending:* Fig. $M5$. Let the plate be supported at A and B and loaded at C and D. The resultant travel of the load D is half from D to A and half from D to B. But as AB and DC are by symmetry zero shear planes half the load at D travels thru the material of the triangle DOA and half thru DOB. If the plate is uniform and of medium thickness it is warped in such wise that AD and DB are unbent and ef and gh are straight line elements of the warped surface. Consequently the vertical shear must be uniformly distributed along eO and the travel of the center of gravity of the load is DkA thru the center of the area of the triangle. By anti-clockwise twist of vertical elements the load traveling from D to A is first moved from the plane AD to Dk thereby distributing the vertical reaction uniformly along eO. At eO the moment passes thru zero and changes sign and the twist about the vertical element changes from anti-clockwise to clockwise rotation, moving the effective shear back to the direct line of travel. In other words, in the area of negative bending anti-clockwise twist has moved the half load at D the distance $L/8$ in its travel in the direction of OA while the clockwise twist in AOe moves it an additional distance $L/8$ along OA but in the direction Ke. The lever arm of effective bending is thus reduced to the distance $AO/2$ and the deflection at O is therefore one half as great as though the loads at C and D were applied at O.

It thus appears that the interference of equal moments at right angles to each other whether of like or unlike sign reduce the numerical magnitude of the effective couple one half, while doubling the number of supports again halves the effective moment in synclastic curvature.

The experimental proof in anticlastic curvature presents the following unique characteristics. If the thickness of the plate be of the order of one twentieth to one fortieth of the length so that the applied loads may be large in comparison to the dead load weight, satisfactory accord in results is observed. If the plate be very thin the curious phenomena of Art. 2 will be observed but the ratio of half deflections maintained. Again, if the order of thickness be approximately one over two hundred and there be some residual strain in the plate as commonly found in celluloid then a degeneration into cylindrical curvature about either axis may occur.

12. The Applied and Resisting Moments Parallel to the Sides of Uniformly Loaded Square Panels Supported on all Four Sides:

Let W be the total uniform panel load then the total uniform load on $AOBD=W/4$ and so far as statical moments are concerned it may be taken to act at the center of $AOBD$. This load has a statical moment of $WL/16$ about any axis parallel to either side when taken with the equal reaction of the corner support in case L is the length of the side, and each of these two independent applied moments at right angles to each other must be independently held in equilibrium by resisting moments in those portions of the beams and slab that are in this quarter panel.

In order to find out just what these resisting moments are the distribution of the load upon the beams as well as the shearing stresses in the panel must be considered. The load upon the beams is uniformly distributed, viz., $W/8$ upon AD and BD respectively, thus producing a bending moment of $WL/32$ in each side beam. Each beam consequently resists one half the total applied moment in that direction in $AOBD$.

Now consider the manner in which the quarter panel of the slab transfers the load to the beams by the shears and moments produced in the slab. It is evident by symmetry that OA, OB and OD are lines of zero vertical shear at all points. In fact, every vertical plane through O cuts the slab in a section of zero vertical shear at all points in case the side beams are uniformly loaded, as they must be provided the stiffness of the slab is small per unit of width compared with that of the beams.

Subdivide AD and BD each into the same number of equal segments and join these points of subdivision with O. Then the elementary triangles so formed are equal to each other in area, and if any line as PE be drawn parallel to AD the vertical shear in each of the segments of PE is equal to the load on the triangular area between it and the two sides radiating from O, because these sides being lines of zero shear, the whole load on the triangle must be supported by the shear on that segment of PE which constitutes the base of the triangle in which alone a vertical force is applied in the form of a shear. This subdivision into elementary triangles consequently transfers the total load $W/8$ which is situated upon the triangle AOD to the beam AD and distributes it there uniformly along AD and in so doing calls into play a moment $WL/48$ which is the product of the load $W/8$ by the distance $L/6$ between the center of gravity of this triangular load and the beam AD which supports it.

In order to consider more in detail this transfer of the load $W/8$ from the center of gravity G of AOD to C its center of support on AD along which $W/8$ is uniformly distributed, introduce two equal and opposite vertical shears $\pm W/8$ at K the one third point of AD. Then $AC=L/4$, $AK=L/6$, $KC=L/12$, and $GK=L/6$.

In transferring the load $W/8$ from G to AD, the resultant of all the vertical shears on planes like PE parallel to AD is a vertical shear $= \pm W/8$ at G, and the resultant of all the vertical supporting shears at the edge of the beam AD is a vertical shear $= -W/8$ at C. Now introduce at K the equal and opposite shears $= \pm W/8$ which are in equilibrium with each other and are parallel to the resultants at G and C. These two shears so introduced at K will not disturb the equilibrium already existing and may be considered in combination with the forces previously enumerated.

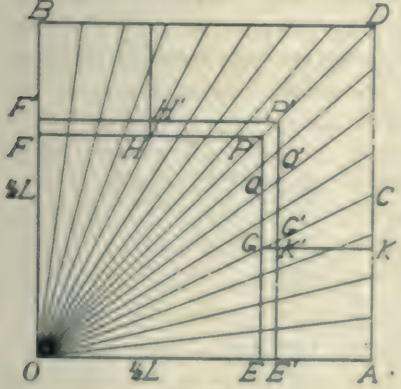


Fig. 25

Take the couple of shears $+W/8$ at G and $-W/8$ at K. They produce a moment $WL/48$ about any axis parallel to AD, which is a bending moment because the shears are in vertical planes parallel to the axis of moments. This is the bending moment in the slab which has been already obtained.

Again, take the couple of vertical shears $+W/8$ at K and $-W/8$ at C, a couple of shears both of which act in the vertical plane thru AD, and which therefore constitute a twisting moment about an axis perpendicular to AD of an amount equal to the product of $W/8$ by $L/12$ viz. $WL/96$. Hence the loads upon AOD and BOD together produce a bending moment of $WL/48$ and a twisting moment of $WL/96$ or a total of $WL/32$ about each of two axes parallel respectively to the sides AD and BD. These bending and twisting moments are resisted by the material of the slab, and serve to transfer the load to the beams so that of the total applied statical moments $WL/16$ about each side, one half is resisted by bending moments in the beams, one third by bending moments in the slab and one sixth by twisting moments in the slab.

Since every element of load on a plate produces some twisting moment it appears that twisting moment is an essential constituent of "plate action" as distinguished from "beam action".

The measure of the twisting moment in this combination of beams and slab is one sixth of the total statical applied moment of the load or one third of the total statical moment applied to the slab.

Since the secret of slab action resides in the twisting moments it is desirable to further consider the manner in which these moments arise: In order so to do, consider first the twisting of a bar or shaft of rectangular cross section subjected to a uniform twist from end to end by the application of equal and opposite twisting moments at its two ends respectively. It is known that the greatest shearing stresses in the right cross sections of such a shaft are found at the middle of the longer sides and that they act parallel to those sides, altho at first sight it would seem as if the greatest intensity of shear would occur at such points as are at the greatest radial distance from the axis of twisting. Such, however, is not the fact because it is the relative rigidities of the several parts of the rectangular cross section which determine the distribution of the resistance to twisting. It is evident that the total resistance to shearing distortion between two neighboring rectangular right sections of a wide bar is much greater along the long side than across it, for any shearing distortion can be produced in the thickness of the bar more easily than in its breadth, whether it be simple shear or a twist about the axis. Hence it appears that the intensity of the distribution of the shears in the right section of a rectangular bar is really controlled by the relative shearing rigidity of its parts and not by the relative distance of these parts from the axis of twist as might at first be supposed. The relative intensity of these maximum shears on the long sides is dependent upon the relative length and breadth of the rectangular cross section, being larger the greater the relative length of the section compared with its breadth.

This admitted experimental and theoretical result is intimately related to Kirchhoff's theorem as to the conditions of equilibirum at the boundary of a plate subjected to external forces, which theorem states that when a twisting moment of given intensity per unit of width of section of a horizontal plate is produced by vertical shears in that section or bounding surface the same twisting moments may be otherwise produced by applying instead equal and opposite horizontal shears in the same bounding section, and any difference in effect between the action of twisting moments of equal intensity produced in these two different ways will not extend more than an infinitesimal distance from the section into the plate.

Now let a vertical plane $P'E'$ parallel to AD pass close to PE, then the magnitude of the total twisting moment acting between PE and $P'E'$ about AO is the product of the total vertical shear S across PE taken as acting at G muliplied by the distance $G'K'$ by which it is shifted along PE in being transferred to G'. Since the shift parallel to AD of the vertical shear on opposite faces of any one of the elementary parallelograms that make up the area $PEE'P'$ increases uniformly from E to P, the intensity of the twisting moment about AO applied to PE increases uniformly along EP from zero at E on the median line to P on the diagonal. In this reasoning the section PE which is employed has no reference to the center of gravity G, but is a section drawn parallel to AD anywhere between O and AD. Moreover the total twisting moment on PE increases uniformly with its distance from O, *i. e.* is proportional to the length of PE itself.

Now the application of the facts mentioned, respecting the twisting of rectangular shafts and Kirchhoff's theorem on boundary conditions, to flat plates or slabs where the twists are produced by the shifting laterally of vertical shears is this:—

In a vertical rectangular section of a plate in which the thickness is so small compared with the breadth as it is in a floor or slab all the shearing resistance to a twist in the section practically consists of horizontal shears of opposite sign above and below the neutral surface which increase from zero intensity at that surface to their greatest positive and negative values at the upper and lower faces of the slab, and this notwithstanding the fact that these twists are produced by the shifting laterally of vertical shears between this and parallel sections.

Having thus discovered the distribution of the twisting moments in their action upon any elementary prism of the strip $PEE'P'$ whose horizontal cross section is $PQQ'P'$ for example, which extends vertically thru the thickness of the plate or slab let us consider the action upon this prism of the applied bending moment and twisting moment somewhat more in detail.

Let $+H$ and $-H$ be the horizontal shears that act parallel to AD on the vertical edges of a horizontal lamina $PP'Q'Q$ one unit thick. The intensity of these stresses $\pm H$ depends upon the distance of this lamina above or below the neutral surface; those above the neutral plane will tend to twist the prism in one direction, while those below it will tend to twist it in the opposite direction, the entire system of shears being held in equilibrium by the twisting resistance of the prism about its vertical axis. There are no vertical or horizontal shears upon the vertical faces PP' and QQ' since these lie in planes of zero shear, but there are opposite normal stresses $+T$ and $-T$ upon the faces PQ and $P'Q'$ of different signs above and below the neutral axis which are due to the bending moment. Since $+T$ and $-T$ altho parallel are not along the same straight line, those acting on every lamina above the neutral plane will produce a twist about a vertical axis which will be held in equilibrium by an equal opposing twist due to the forces acting on corresponding lamina below the neutral plane. The above mentioned twists and any twists due to normal stresses on the vertical faces PP' and QQ' are resisted by twisting shears in the neutral plane and planes parallel to it, whose intensity decreases with the distance of the plane from the neutral surface and they call for no resistance elsewhere.

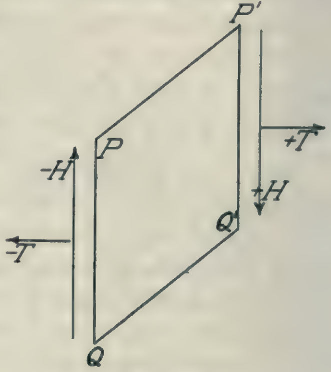


Fig. 26
Horizontal Lamina or element of vertical rhomboidal prism extending thru the plate.

From this explanation and exposè it is evident that the horizontal shears on the vertical sides of a prismatic element extending thru a plate have no necessary relation to the horizontal shears on vertical planes at right angles to them since they each merely produce independent self-contained twists in the prism, and the proposition which has been used in establishing the accepted fundamental differential equations of plates to the effect that the intensity of twisting moments is the same on any two vertical planes that cross at right angles is not necessarily true since it is based on incorrect assumptions which omit to consider twists about the vertical.

12b. The Applied Bending and Twisting Moments About the Diameter of a Circular Cantilever Plate Restrained at its Center:

(*1*) *Uniform load thruout.* Let $AOBD$ Fig. 27 represent one quarter of a uniform circular plate of radius $= R$ restrained at O.

Let W be the uniformly distributed load upon the entire plate, then the total uniform load upon $AOBD = W/4$, and so far as moments about AO or BO are concerned, this load may be taken to act at C the center of gravity of this quadrant at a distance $4R/3\pi = 0.424 R$ from AO and BO. This load when taken with the equal reaction of the central support has a statical moment $= WR/3\pi$ both about AO and about BO independently.

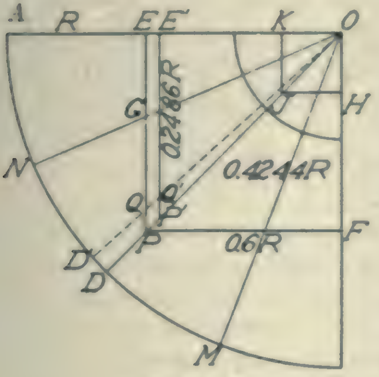


Fig 27

Each of these applied moments must be held in equilibrium by resisting moments in the material of this quarter of the plate, and in order to determine in more detail how this is accomplished it may be noted that all diametral sections thru O cut out by radial planes are sections of zero shear at all points of these sections, and that the load is transferred by vertical shears to O along radial lines only. For example, the load upon any area $ADPE$ is carried by the vertical shear in the section PE between it and O. Now let D be the middle point of the quarter circumference AB. Then the load $W/8$ upon the octant ADO may be regarded as transferred to O by shear in sections of ADO parallel to BO and by the moments that these shears produce. Let G be the center of gravity of the load on the octant ADO, then $OE = 4\sqrt{2}R/3\pi = 0.6R$, and $GE = OE$ tan $22°30' = 0.25R$ nearly.

The resultant of all the shears in ADO may be taken, so far as the moments produced by them is concerned, as a resultant shear $W/8$ acting at G the center of gravity of ADO in the section PE parallel to BO, while the reaction of the support which supports this part of the load acts either at O or near by at some section JK of the column cap parallel to BO. At E introduce two equal and opposite vertical shears, $\pm W/8$ in the plane PE which will in no way disturb the action of other shears. Then the vertical shear $-W/8$ at O, taken with $+W/8$ at E acting parallel to BO will produce a moment about OB of $OE \times W/8 = 0.6\ R\ W/8$, which is a bending moment because the shears producing it act on planes parallel to the axis of moments OB.

Again, the vertical shear $-W/8$ at E and $+W/8$ at G acting in the plane PE will produce a moment about OA of amount $GE \times W/8$

$=0.25\ R\ W/8$, which is a twisting moment about AO, because the shears producing it both act in the plane PE at right angles to the axis of moments. These two moments and the shear $W/8$ at G parallel to OB transfer the load in this octant to O. Likewise the shear $W/8$ in BDO acting in vertical planes parallel to AO transfers the load on the octant BDO to O by help of bending moment $=0.6\ R\ W/8$ about AO, and a twisting moment about $BO=0.25\ RW/8$.

The total statical moment about AO or BO is the sum of the bending and twisting moments about either axis, viz.

$$(0.6+0.25)\mathrm{R}\ W/8=0.425\ R\ W/4$$

which is the same amount as was obtained when the quadrant was treated as a whole. But this detailed analysis has established the fact that the statical moment about each of the independent axes AO and BO at right angles to each other, viz. $0.424\ R\ W/4$, is made np of two parts, only one of which is a bending moment, viz. $0.3\ R\ W/4$, so that the bending moment is only 70% of the total statical moment, while the twisting moment which measures the slab action amounts to 30% of it, in case the reaction of the support is concentrated at the point O. But in case the reaction of the support is taken to be uniformly distributed over the area of a circular cap of radius $=R_o$ the bending moment about a diameter will be only $0.3(R—R_o)W/4$, so that when R_o has a minimum value of $0.2R$ the bending moment is only $0.24\ R\ W/4$ or less than 51% of the statical applied moment.

The distribution of the twisting moment along any section PE depends both on the intensity of the vertical shear along PE and on the shift of shear between any segment PQ and the adjacent parallel segment $P'Q'$. This shift is zero at E and increases continuously from E to P because of the decreasing angle at which PE intersects the several radii between AO and AD. The twisting moment along EP reaches its maximum intensity at P on the radius that makes an angle of 45° with PE, and total twisting moment on any section parallel to PE becomes greater the greater its distance from A where it is zero. It is evident therefore that the intensity of the twisting moment along any section that is a chord of a circular cantilever plate, is zero at the middle of the chord, and increases along the chord to the point where it intersects a radius at 45° with it after which it diminishes to zero at the edge of the plate.

(2) Uniform load around the circumference: Let O be the center of the plate and ADB be one quarter of the loaded circumference which carries a load $W/4$, whose center of gravity is situated at a perpendicular distance of $2R/\pi=0.6366R$ from either AO or BO, so

that its statical moment about AO or BO is the product of $W/4$ by this distance, viz, $RW/2\pi = 0.6366RW/4$.

Now consider the separate loads $W/8$ upon the two octants AD and BD. The center of gravity of a uniform load on AD is situated at perpendicular distance of $2R\sqrt{2}/\pi = 0.9R$ from BO, and at a distance of $0.9R \tan 22°30' = 0.373R$ from AO so that the load $W/8$ upon AD has a moment $= 0.9RW/8$ about BO and a moment $= 0.373RW/8$ about AO.

The first of these moments is a bending moment and the other a twisting moment. Again, when we consider the load $W/8$ upon the area BD it produces a bending moment and a twisting moment about AO and BO respectively of the same magnitudes as those just computed; so that the total moment about either AO or $BO = (0.9 + 0.373)RW/8 = 0.636RW/4$, i. e. the same as that previously calculated. This twisting moment is more than 41% as large as the bending moment.

It is easy to find the position upon ON of the center of gravity of a composite load on AOD which is made up of one part uniformly distributed over the area of the plate and another part uniformly distributed around its circumference in case the relative magnitudes of these parts are known. It is evident that the magnitude of the twisting moments produced by such a composite load about any radius is more than 40% of the bending moment about that axis, since that is the fact respecting each of the component loads separately. The total statical moment about the axis is the sum of the twisting and bending moments taken together.

13. Statical Demonstration of the magnitude of the bending moment parallel to each edge of a square panel of a column supported flat plate. In Fig. 26 let O be at the middle of a panel and D at the center of a column cap of which HJK is the quarter of the perimeter lying within the quarter panel $OADB$.

In case of a total uniform load of W pounds per panel of a continuous uniform plate of many panels, since by reason of symmetry the diagonal OD as well as all four sides of $OADB$ are lines of zero vertical shear at all points, the entire load S resting upon the trapezoidal area $OAHJ$ is wholly

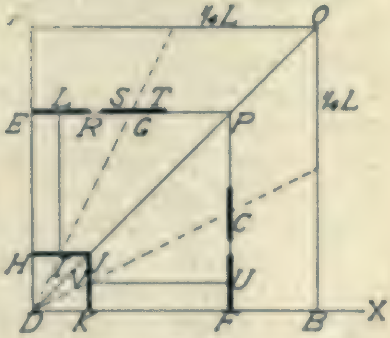


Fig. 30

supported by the vertical shears at the edge *HJ* of the cap, and the equal load *S* on the area *OBKJ* by the shear at the edge *KJ*.

Similarly the vertical shear at any section *PE* parallel to *OA* is equal to the load upon the area *OAEP* lying between *PE* and the median line *OA*.

Let *G* be the center of gravity of the area *OAHJ*, then *DG* bisects both *PE* and *OA* and all sections parallel thereto, and the resultant shear *S* due to the entire load $W/4$ upon *OAHJ* acts at *G*, but is transferred from *G* and applied to *HJ* by help of the statical moments produced in the plate by this transfer.

In order to determine precisely what these moments are, assume lengths *ST* and *ER* equal and parallel to *HJ*, and for convenience let the resultant shear *S* at *G* be uniformly distributed along *ST*. Also assume that two equal and opposite shears equal to *S* act on *ER*. They will in no way disturb the equilibrium already existing because they themselves constitute a system in equilibrium and may consequently be arbitrarily introduced into the system previously existing for the purpose of demonstration, if so desired, without affecting results in any way.

Now designating upward shears applied to *OAHJ* as negative and downward as positive the shears and moments acting on *OAHJ* may be enumerated in pairs, as follows:

1st. A negative shear *S* at *HJ* and a positive shear *S* at *ER*, these together transfer the reaction *S* of the support from *I* to *L* and in doing so produce a bending moment $=S\times IL$ about *OA* or *BD* or some other axis perpendicular to *IL*.

2nd. A negative shear *S* at *ER* and a positive shear *S* at *ST*, these together transfer the supporting reaction from *L* to *G* and hold the resultant load *S* in equilibrium at *G* and in so doing produce, (since they are equal and opposite shears in the same plane) a twisting moment $=S\times LG$ about *AD* or *OB* or some other axis perpendicular to *LG*.

Similarly the load upon *OBKJ* produces a positive resultant shear *S* at *C* the center of gravity of this load at the mid point of *PF*, and this with the negative shear *S* at *KJ* produces a bending moment $=S\times VU$ about some axis as *AD*, perpendicular to *VU* and a twisting moment $=S\times UC$ about some axis as *BD*, perpendicular to *UC*.

But $LG=\frac{1}{2}VU$, and $UC=\frac{1}{2}IL$. Hence the twisting moment about either axis perpendicular to a side of the panel is one half as great as the bending moment about the same perpendicular, and the load *S* upon each of the trapezoidal areas *OAHJ*, *OBKJ* produces a bending moment about an axis parallel to one side of the panel and a twisting moment about an axis perpendicular to that side.

In case the column cap is reduced in size to a mere point at D each trapezoid becomes a triangle of area $= \frac{1}{4}L^2$, and $LG = UC = L/6$, while $IL = VU = \frac{1}{3}L$. Hence the bending moment M_x reduces in this case to $WL/24$ and the twisting moment to $WL/48$. With the ordinary size of column caps the moment each way reduces to $WL/16$ and sometimes with a large cap to as low as $WL/20$ as will later be developed. Similar results hold true in case the panel is not square.

Remarks: The total statical moment about a side for the entire load on a half panel between a median line and a parallel panel edge is twice the sum of both the bending moment and the twisting moment for a quarter panel. This sum amounts to $WL/8$, which statical moment must be held in equilibrium by the resistance of the combined bending and twisting moments that act in the material of the plate about the side in question.

14. Equilibrium of an Elementary Vertical Prism Under Bending Moment and Twisting Moment. If in Fig. 30. on a previous page, AO be divided into equal parts and radial lines be drawn from the point of support D to each of these points of division as in Fig. 31, each pair of consecutive rays will bound a triangular area such as N_1DA, N_2DN_1, etc., having a common height and base and therefore of equal area though different in configuration.

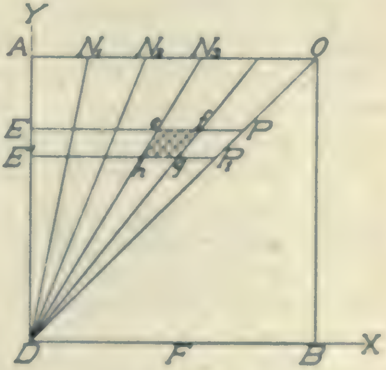


Fig. 31

Each of the radial sections thru D is a section of zero shear at every point and the load upon any trapezoidal area such as that between ef and the corresponding segment of AO produces a vertical shear at ef equal to that load.

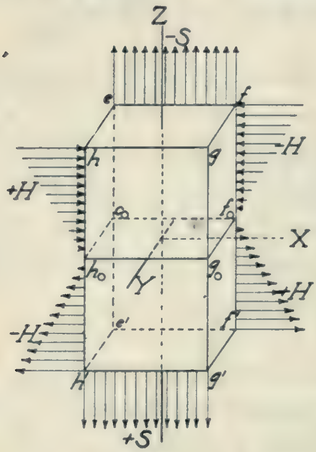


Fig. 32

Designate its amount per unit length of *ef* by $+s$, then the downward applied vertical force exerted on the element *efgh* at *ef* by the load between it and *AO* differs by no more than an infinitesimal amount from the upward supporting force $-s$ exerted on it at *gh* by the triangular portion of the plate between *gh* and *D*. These forces are vertical shears of intensity $\pm s$ which are uniformly distributed along *ef* and *gh*.

It is to be observed that these shears on *ef* and *gh* are not perpendicularly opposite to each other, but on account of the obliquity of the rhombus *efgh* one shearing force is shifted laterally with respect to the other, at the same time that it is shifted perpendicularly to the plane on which it acts. The perpendicular shift produces bending moment, and the lateral shift twisting moment as may be seen from Fig. 32, where the vertical shearing resistance which the element *efgh* exerts at its back face *eff'e'* on the trapezoid between it and *AO* is taken as equal and opposite to that at the front face *ghh'g'* on the triangle *Def*. In this Fig. 32 it does not appear so clearly as on Fig. 31 that these forces are not directly opposite each other, but that may nevertheless be readily understood.

Vertical shear is uniformly distributed along any section *PE* parallel to *AO* as may be seen from the fact that the shears on equal segments are produced by equal portions of the load. It should be further noted that since the twisting moment depends upon the lateral shift of this uniform shear it increases uniformly from zero at *E* to its greatest value at *P* at the diagonal *DO*.

Now the twisting moment produced by this lateral shift of the vertical shears between front and back faces about a horizontal axis *Y* is such as would tend to cause the laminae parallel to front and back faces to twist around on each other about *Y*. But the couple furnished by horizontal shears $\pm H$ on the back face, and horizontal shears $\mp H$ on the front face will cause the same kind of twisting about *Y*. As already explained these horizontal shears in vertical planes are the effective resisting shears in plates so far as twisting moments about horizontal axes are concerned.

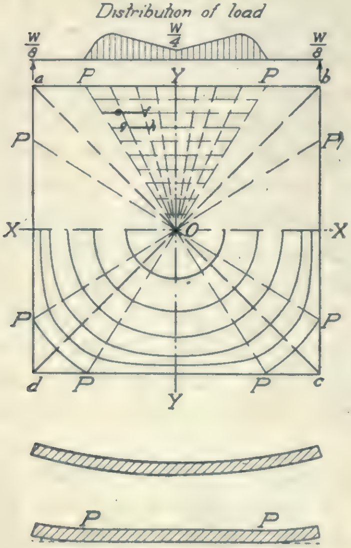
It will be observed that the horizontal shears $\pm H$ on that part of the front and the back face that is above the neutral surface furnish a twisting moment about the vertical axis Z that increases in magnitude from the top surface to the neutral surface, etc., and further that it is held in equilibrium by the opposite twisting moment produced by $\mp H$ below the neutral surface. These moments tend to make the prism twist about the axis Z. So far it does not appear that there is any necessary connection between these horizontal shears on vertical planes and the vertical shears on the same planes.

When the intensities of the shears are equal on vertical planes at right angles to each other they have equal and opposite twisting moments about a vertical axis and neutralize each other. But this need not take place. As already seen on a previous page twisting moments about a vertical axis may also be produced otherwise than by these shears, viz. by the lateral shifting of the longitudinal bending stress T. The relative amounts of resistance to the resultant twisting moment about a vertical axis per horizontal unit of area are adjusted between unequal horizontal shears on vertical planes at right angles and the twisting shears on horizontal planes by the relative rigidities of these two kinds of resistance. And it is by means of this adjustment that the longitudinal stresses of the bending moments are differently distributed in different cases. For example in the crown sheet of a boiler which is pierced for stay-bolts at points of support the stresses across the sides of the panels are different from the stresses in a plate not so pierced.

It is to be noticed that the preceding discussion does not require the intensities of the shearing components of the stresses on vertical planes which are at right angles to each other to be equal, because of the intensity of the twisting moments that may exist on horizontal planes. In this particular the equilibrium of a rectangular elementary prism extending from top to bottom of a flat plane differs from that of a horizontal prism extending thru from side to side of a beam, which last is so symmetrically situated with respect to the shearing stresses that no twisting moments can occur in it as they can occur in an element of a flat plate. For this reason, vertical and horizontal shearing stresses have equal intensities in simple beams but horizontal shears on vertical planes at right angles need not be equal in flat plates.

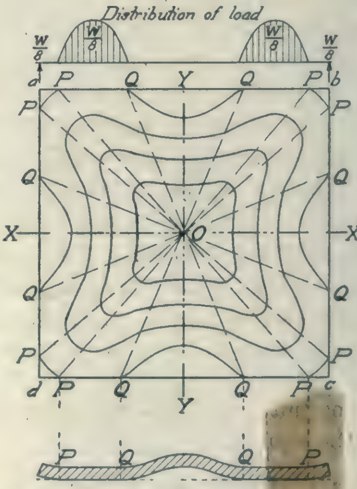
15. Warping and Twisting of Flat Plates. Let us consider the action of a single concentrated load W applied at the center O Fig. 98 of a thin square plate which is freely supported upon but not fastened to four rigid side beams that extend from column to column. Let the load be large enough to produce a center deflection of one or two percent of the span L. Under such deflection the corners of the plate would rise from their supports. The statical moment of the reactions at two supports like a and b on a single side about a median line XX of the plate would be $\frac{1}{4}WL$. Since the plate curves upward from the beams at and near its corners, the applied load is distributed unevenly along the beams and the plate will remain in contact with them only along a middle segment in each one extending from mid span each way to some points PP on each beam where the distributed vertical pressure will have zero intensity. Starting from any one of the points P the intensity of the load applied to the beam will therefore increase from zero at P to a maximum between P and mid span. There will be eight of these points P, two on each side beam, and radial sections drawn from the center O to each of them will be sections of zero vertical shear, in the panel. Any other vertical sections whatever thru O will also be sections of zero vertical shear.

But the area between any such line OP and the nearest median line XX or YY, is one in which the total vertical shear on any section perpendicular to that median line amounts to $W/8$ on that part of the section between OP and the median line. The mean intensity of this vertical shear decreases proportionately to its distance from O, because the constant total amount of the shear on each vertical



Side view under moderate concentrated load

Fig. 98



Side view under heavy concentrated load

Fig. 99

section within the triangle POP amounting to $\frac{1}{4}W$ is spread over a length of section which increases uniformly with its distance from O. It thus appears that the upward reaction that the segment PP of the beam ad exerts upon the plate is $\frac{1}{4}W$. Its moment about YY amounting to $WL/8$ is a bending moment. This moment comprises only one half of $\frac{1}{4}WL$, the moment about YY due to the reactions at a and d, and is resisted by bending resistance in the plate itself.

It appears therefore that while $\frac{1}{2}W$ is transmitted to a and d together, and $\frac{1}{4}W$ by way of the beam ad, $W/8$ is transmitted to a by way of the beam ab, and $W/8$ to d by way of cd. Consequently the load $W/8$ which is carried by vertical shears on sections parallel to ab and applied to the beam ab along the segment PY of it between a and Y has a center of application between P and mid span, so that the moment applied to bend the beam is less than $WL/16$ by reason of the fact that its arm is less than $\frac{1}{2}L$. The amount by which the moment resisted by the beam is less than $WL/16$ is that of the twisting moment in the slab about YY called into play by applying the load $W/8$ which originates at O to the beam ab at a distance from a less than $L/2$. This twisting moment and the bending moment in the beam ab together amount to $WL/16$.

The importance that attaches to the present discussion of the action of a concentrated load on a square plate arises from the fact that it is here shown how the load is distributed and applied to the side beams by means of bending and twisting moments in the plate, and that the distribution of the load along the supporting beams is due solely to the twisting moments, which as will be seen later, are also principally responsible for the warping of the plate. It is further shown that whatever may be the magnitude of the twisting moments they reduce by just so much the total amount of the applied bending moment below the statical moment of the applied forces due to the loads. Hence it is evident that if twisting moments are called into play in distributing a single concentrated load upon the supports, they are also necessarily called into play by any distribution of loads, uniform or otherwise, since these may be regarded as made up of a combination of concentrated loads placed more or less closely together.

The twisting moment in that part of a section parallel to ab which is on one side of YY is resisted and held in equilibrium by an equal and opposite twisting moment on that part of the section on the other side of YY thru horizontal shearing resistances of opposite signs above and below the neutral surface. These horizontal shears are zero and change sign at YY and increase in intensity to OP where they are greatest. They are also resisted by those portions of the same sections between OP and the outside edges of the plate. It is the deformation produced in these last mentioned portions by the horizontal shears that is the primary cause of the curling up of the plate near the corners.

The magnitude of the twist about an axis lying in an initially flat plate is measured by the difference of slope produced between the tangents to two normal sections of the neutral surface, which are made by planes perpendicular to the given axis drawn at a distance of one unit apart along the axis.

Now the distribution of the load $\frac{1}{4}W$ which is carried to ab and applied along PP by the vertical shears on sections parallel to PP may be represented by drawing a series of radial sections of zero shear between OY and OP such that they cut off distances along PY to which equal shears are applied. The segments are shortest where the intensity of the beam load is greatest and the segments become longer the nearer they are to P. The twisting moment about OY is formed by elements due to vertical shears which are shifted parallel to ab in transmitting an element of the load toward ab. For example the element of vertical shear acting on ef is equal and opposite to that on gh. The product of that element of shear by the perpendicular distance between ef and gh is an element of bending moment about XX, but the product of it by the distance it is shifted parallel to XX is an element of twisting moment about YY. It is evident that the twisting moment vanishes at YY on sections perpendicular to YY, and reaches a maximum of intensity at PO in such sections.

The twisting moment applied to the faces ef and gh tends to produce right hand distortion or angular motion along OY from O toward ab with horizontal shears of the same signs in prolongations of ef and gh toward ad. Such a distortion makes the prolongation of gh toward OY slope downward and makes the prolongation toward ab slope upward so as to raise the plate off from the beams near a. The slopes produced in sections near to ad and parallel to it also cooperate to the same effect.

The contour lines in this plate are shown in Fig. 98, where they differ from those of a circular plate carrying a concentrated load at its center and supported around its circumference in that the contour lines near the sides are flattened out so those that are nearest become almost parallel to the sides near mid span. Those further away from the sides are not so much changed from a circular form. The deflection curve for a mid section is shown immediately below the plan view and the side elevation after bending below that.

The foregoing analysis is applicable to plates having a thickness of the order of 1/40th of the span. For a plate whose thickness is of the order of 1/400th of the span length, this analysis would not be applicable. The difference in the mode of operation of a very thin plate of homogeneous material and a thicker plate under a concentrated load at the center brings out clearly the fundamental difference between plate action and beam action.

Support a sheet of celluloid 1/16″ thick, 12″ square, freely along its edges at about ¼″ in from the edge. Set a block 2″ in diameter in the center of the plate and apply a gradually increasing load on this block. Under a load of 6 or 8 pounds it will be found that the corners of the plate rise slightly from the supports and the contours are approximately those shown in the preceding figure for Type III floor. When this load is gradually increased to 16 or 20 pounds, a sudden change in the phenomena is observed at some intermediate load, where the plate no longer remains in contact with the supports at and near the middle of the sides but rises from the supports at these points as much as ¼″ or ⅜″ as the load is increased as indicated in Fig. 99. The contours, previously concave toward the center of the panel, now become convex toward the center opposite the middle of the sides, and become star shaped outward along the diagonals. When with increase of load the plate commences to curl up along its median lines at the support the deflection at the load suddenly increases, illustrating the phenomena before explained that as the contours change from convex outward to a form alternately concave and convex or star shaped the circumferential resistance is thereby destroyed and the mode of operation changed. Coaction of circular and radial stress then ceases and deflection increases suddenly with this change. By curling up at the middle of the sides the radial resistance of the plate is increased since the moment of inertia of a corrugation is greater than that of the plane plate. Nevertheless, by the removal of the cooperation of circular ring stress with the radial the deflection has increased although the corrugations formed in the plate have increased the radial resistance to bending. Twisting moments also have increased at the same time, since they throw the load carried by the side supports nearer to the corners of the panel and distribute it as shown at the top of Fig. 99. No load, of course, can be brought upon a side support at points at and near the middle of a side where the plate has risen from it, and each half length of a side beam has a loaded segment *PQ* at and near each of its quarter points. The triangular area *QOQ* has zero vertical shear in all sections parallel to *QQ*. The only vertical shears are in the eight triangles *QOP*.

The action of such a plate shows the absurdity of attempting to treat a flat plate supported on four sides by beam strip theory with strips parallel to the sides since the load not only does not travel to the supports in accordance with that theory but is carried in a large measure towards the corners of the plate by twisting moments induced in the plate by double bending.

16. Stress Analysis: Square Plate, Anticlastic Bending. Let *S*:*S′ PP′* Fig. *P*3 represent a square plate supported at opposite corners *S* and *S′* and loaded at the other corners *P* and *P′*. Suppose the plate of such thickness that the weight of the plate is so small compared to the loads P, P_1 that the curvature may be treated as that caused by the load, i.e., affected by the dead weight of the plate itself to a negligible degree only.

After loading the plate will be bent in anticlastic curvature. The principal curvatures along the diagonals will be parabolic. Along *XX* and *YY* in the vertical plane the curvature will pass thru zero. Each horizontal plane section will cut a hyperbola from the top and bottom surface whose major and minor axes lie in the respective vertical planes passing thru the diagonals of the plate. The surface is accordingly a hyperbolic paraboloid.

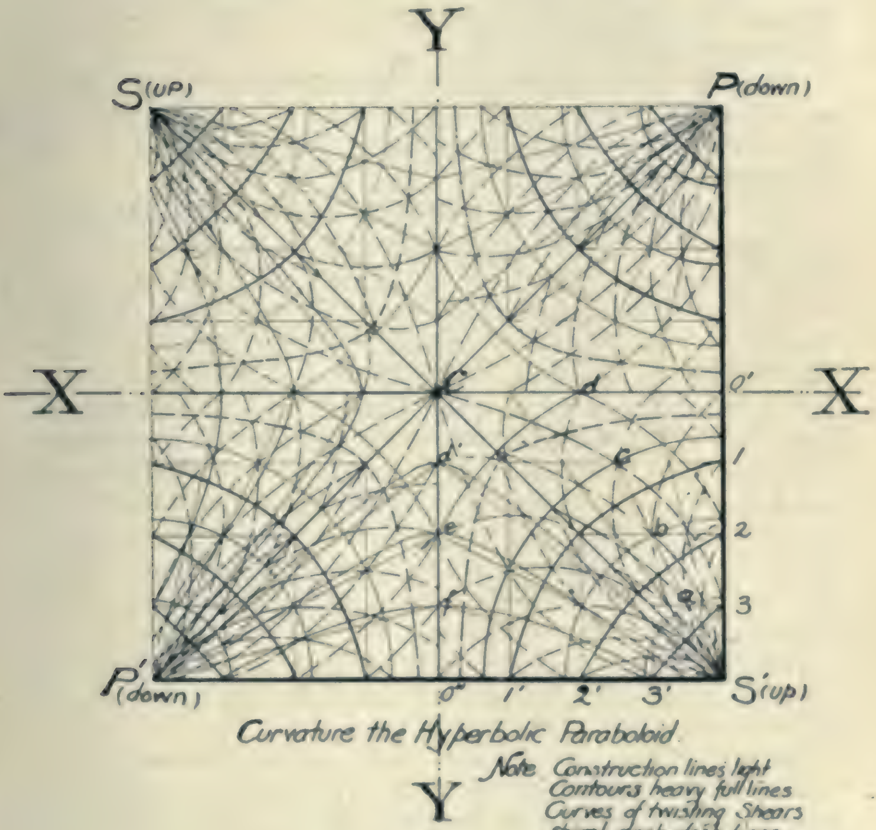
The travel of the center of gravity of the load will be that indicated in Fig. *M*5. Thru equal divisions of its path *abcd* Fig. *P*3, draw circular coordinates to the boundary. Divide *CO′* and *CO″* into four equal parts and draw radials from *S′* which represent equal increments of shift of shear by torsion. Divide the half span *O′S′S′O″* into four equal divisions and draw parallels to *XX* and *YY* from the boundary to the diagonal.

Thru the intercepts of the radials and circular coordinates draw right and left spirals and in like manner thru the intercepts of the radials and parallels to *XX* and *YY* at 1,-2,-3, 1′,-2′,-3′.

The spirals will then represent coordinated curves of greatest horizontal shear in vertical planes.

The surface is a true warped surface which may be generated by a line as *XX* moving normal to *YY* and touching or directed by *S′ P* and *P′ S* which are straight lines of opposite inclination in planes parallel to *YY*.

FIG P 3.
Square Plate Anticlastic Bending.



Curvature the Hyperbolic Paraboloid.

Note Construction lines light
Contours heavy full lines
Curves of twisting Shears
Spiral dash-dash lines
Intermediate Contour between
1-1' and 0'C dash-dash line

Contours accordingly intersect the boundry S' P' S' P', etc.' at equal intervals and across the corners of the plate their direction will be determined by symmetrical intersections of the shear spirals.

Since the principal curvature of the plate along the diagonals is parabolic the intersection of the contour curves with the diagonal may be verified at once by computation; thus coordinating the graphic method with that of the mathematical theory presented in Natural Philosophy by Thomson and Tait, or the works of

Love or Ibbetson, dealing with the mathematical theory of elasticity.

The tension and compression faces of the plate present this difference in appearance in that the curves of greatest angle intersect the diagonals on the tension face at a more acute length than on the compression face.

The directions of greatest and least displacement make angles of 45° to *XX* and *YY* at *C*, but differ more and more from 45° as the stretch of anticlastic curvature increases from *C* to the boundary along *XX* and *YY*.

17. Stress Analysis of a Circular Plate Supported at Its Periphery: Since a 90° sector only made up of +45° and −45° about *Y* can bend about *X* the coordinates of the twisting shears are fixed by dividing the radius and the 90° sector into the same number of equal parts and drawing the shear spirals as in the lower half of the figure thru the intersection of construction circles 1, 2, 3, and radii 2, *a*, *b*, *c*, *d*, *e*, etc. Draw heavy circles through the shear line intersections between the equally spaced construction circles 1, 2, and 3, and these will be contours of deflection and principal stress lines as well. Every radial will be a line of principal stress. As the configuration of the lines of greatest shear is fixed by symmetry the curve of flexure will be of the same* species whether the load be concentrated or uniform, the deflection varying in proportion to the applied moment.

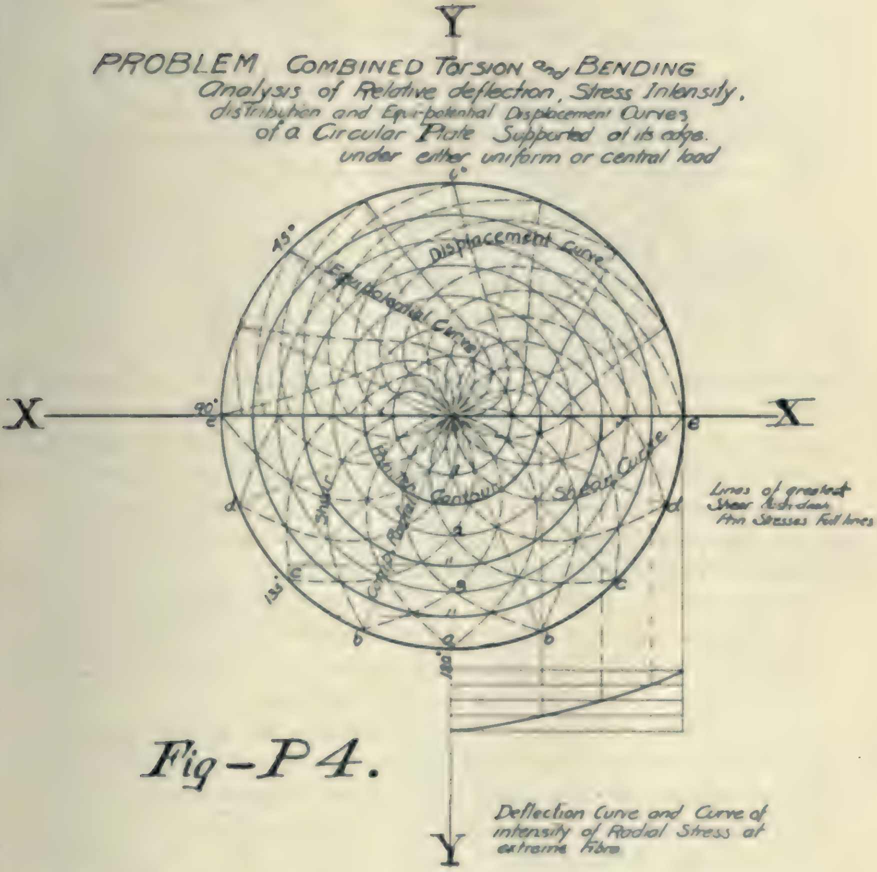
The lower diagram of Fig. *P*4 shows graphically relative stress intensity of the outer radial fibers which by (7) is the relative curve of flexure as well.

Radial bending stresses follow the linear law of increase at mid span from zero at the neutral plane to a maximum at the outer fiber. Now the center of the fiber stress area is .7071 times a half depth from the center of gravity of the section but the center of the shear stress area constituting the reaction to the horizontal fiber stresses is the center of a parabolic area approximately .27 only from the center of gravity at the section. Consequently, equilibrium requires that the neutral plane‡ dip downward from mid span toward the support in order that the center of the fiber stress area may be brought into equilibrium by the center of the shear stress area. The curve of the neutral plane as was shown more in detail in discussing beams thus becomes a parabolic curve for concentrated central load.

Since every radial is an axis of symmetry the twist of vertical

---

* See Love's Mathematical Theory of Elastic Solids.


Y
PROBLEM COMBINED Torsion and BENDING
Analysis of Relative deflection, Stress Intensity,
distribution and Equi-potential Displacement Curves
of a Circular Plate Supported at its edge.
under either uniform or central load
0°
45°
Displacement Curve
Equi-potential Curves
X
90°
e
e
X
Shear
Potential Contours
Shear Curve
Radial
Curves
d
d
c
c
135°
a
b
b
a
180°
Y
Lines of greatest Shear Fuls dashes Prin Stresses Full lines

Fig–P4.

Deflection Curve and Curve of intensity of Radial Stress at extreme fibre

elements above the neutral plane may be either all clockwise or all anticlockwise. The spirals of greatest displacement will will have half as great pitch as the shear spirals and the spirals of least displacement will have twice as sharp a pitch as the curves

---

‡ This conclusion apparently but not really differs from that of Thomson and Tait, Natural Philosophy, 1867, in that the shifting of the neutral plane towards the convex side is not uniform as might be inferred from Art. 633 of Natural Philosophy, quoted as follows: "The whole thickness of the plate remains unchanged at every point; but the half thickness on one side (which when the curvature is synclastic is the convex side) of the middle surface becomes diminished and the other side increased by equal amounts comparable with the elongations and shortenings of length equal to the half thickness measured on the two side surfaces of the plate." The curvature dealt with in the preceding problem differs from spherical or synclastic curvature and the shift of the neutral plane in like manner also differs and the apparent difference in result is merely a difference in the respective problems under consideration.

of greatest shear when drawn by bisecting the angle between the curves of principal stress and greatest shear.

In the case of torsion of the round shaft the circumferential tensions increase as the square of the distance from the center and for analagous reasons the circumferential squeeze of the circular plate as a whole in bending follows the same law.

18. Stress Analysis for a Square Plate Supported at Its Corners A, B, C, D: Central Load at *O*, Fig. *P*5. Divide the half diagonal of the plate into four equal divisions at points 1, 2, 3 and draw normals to an intersection with the median lines. From *O* as a center draw circular coordinates thru the same points. Now the load must travel to the corner supports thru triangular octants such as *O* 5 *A*, etc. The interference of moments causes a twist shifting or distributing the load outward until 2-5 is reached when the reduction of section causes a reverse twist shifting the shear backward toward the direct line *Oa*., i.e., the vertical shear distributes itself thru the total resisting section by the aid of horizontal twisting shears in vertical planes.

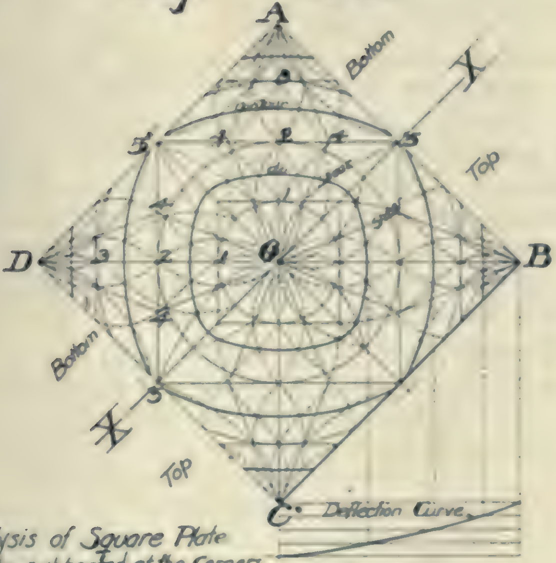
Since there is no restraint at the boundry interfering with the operation of these shears equal increments of load would be shifted or distributed thru equal angles about the foci of radial coordinates *O A*, etc. Hence the vertical shears along 5-5 would have a relative intensity inversely proportional to the tangents of the equal angles of shift and the plate would be bent downward between points 5-5. Contours and curves if displacement are constructed as in the previous example. The solution of this problem tho of value in the clarification of theoretical methods is of little utility in practical construction.

Figure P5b analyzes the same plate for uniform load in the upper half and contrasts the coordinates oft he twisting shears for concentrated load at *O* to the left with those for uniform load to the right in the lower half of the figure.

For the concentrated load at *O* since one eighth of the load must traverse the triangle *O*5*C* in reaching the support *C*, the sign of the increase in section subject to vertical shear changes at 5 and the direction of the twisting shear changes with it.

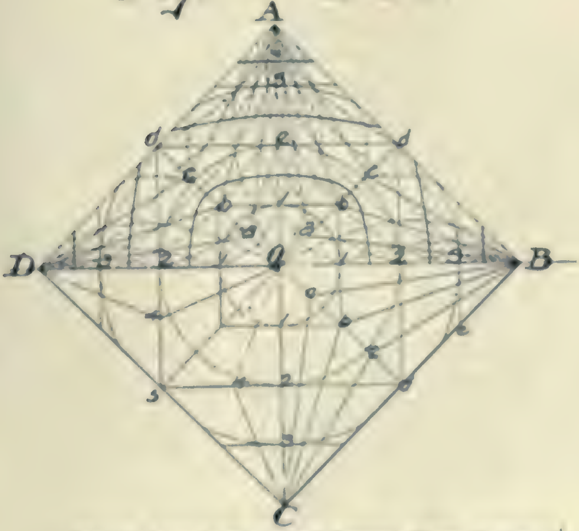
For the uniform load, the twist about the vertical element does not change sign because the straight line travel of the center of gravity of the load to the support traverses the center of gravity of the resisting material without change in direction as it cannot do in the case of the concentrated central load.

Fig. — P5.



Analysis of Square Plate simply supported at the Corners under a Central Load at O.

Fig. — P5b.



Stress Line Analysis of Square Plate Supported at the Corners shown above DB. Coordinates of Twisting shears for Central Load O shown in quadrant DOC and for Uniform Loading quadrant COB

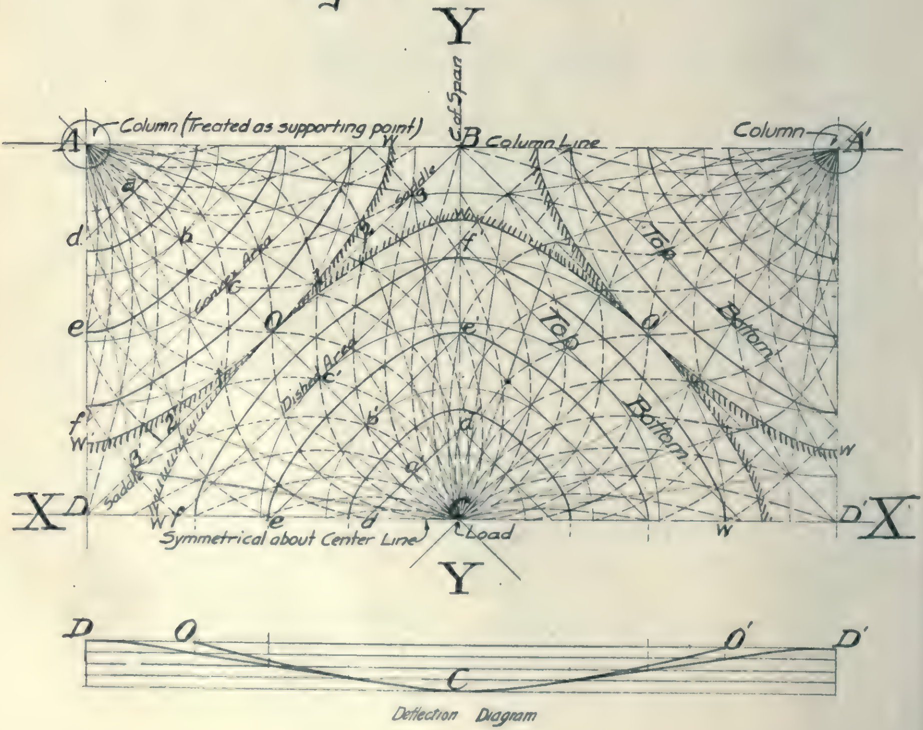
19. Stress Analysis of a Column Supported Continuous Uniform Flat Plate.

Let $ABCD$ be a quarter of a square panel of such a plate CD being the median line on the axis YY and BC being a median line on the axis XX. Assume a concentrated load at the diagonal center of each panel covering an area equal to the area of the column capitals. If the weight of the plate be negligible with respect to the weight of the load, then by symmetry the cantilever area will equal the suspended span area and the line BD is a straight line contour, bisecting the anticlastic area between the convex and concave dished areas.

The principal planes of the dished areas are the diagonals of the panel, i. e., AC of the quarter panel, and of the anticlastic or saddle area YY and AB at the top of the quarter panel and AB and XX at the left.

STRESS LINE ANALYSIS OF A COLUMN SUPPORTED CENTRALLY LOADED FLAT SLAB

Fig P6



By symmetry the curvature of the plate caused by the concentrated load along the column lines and across them on the median lines of the panel is equal but of opposite sign. Hence *DB* at forty five degrees to these lines is the locus of the maximum twist at which the curvature passes thru zero, the twisting shears change sign and along which the vertical shear is constant.

To locate the spirals of twisting shear divide *OA*, *OC*, *OD* and *OB* into four equal divisions, draw radials *A*1, *A*2, *A*3, *C*1, *C*2, *C*3, representing the trace in plan of vertical planes of zero shear. Draw normals to the diagonal *AC* thru the points of equal division *a*, *b*, *c* -*a'*, *b'*, *c'*, to *d*, *e*, *f*, and *d'*, *e'*, *f'*, then will *ad*, *be*, *cf*, *OD*, *OB* in plan represent the trace of vertical planes of greatest distributed twisting couple. About *A* as a center draw circles thru *a*, *b*, *c*, and from *C* as a center draw circles thru *a'*, *b'*, *c'*. These circles are the trace of planes of greatest vertical shear. Construct the shear spirals as in the problem of the square shaft.

Draw curves thru alternate intersections of the shear spirals and these curves will be contours of equal difference in elevation and principal circumferential stress curves at the middle surface of the plate. The shear spirals will differ on the tension and compression faces as noted in the problem of anticlastic bending of the square plate, but lines drawn thru the reverse spiral intersections will be identical curves, i. e., the intersections of the two different series are points on one and the same contour.

The principal radial stress curves will bisect the angles between the curves of greatest twisting shears on the tension and compression faces respectively. But these principal radial stresses in plan will not intersect the contour lines orthogonally except at planes of principal curvature and pure shear, i. e., the column lines and diagonals of the panel and at the straight line contours *OD* and *OB*.

The orthogonal curves of displacement and equipotential are with this suggestion readily drawn bisecting the angle between the curves of twisting shear and the principal stress lines. It will be seen that these curves form a cusp in intersecting *OB* and *OD* at 45° near *B* and *D*. The angularity of the curves of displacement decreases towards *O* where the circumferential compression passes thru a maximum value.

Upon vertical planes of principal curvature stress line analysis may be carried out as shown in Sec. II. for beams.

In considering the straight line travel of the load toward the support in the column supported plate of Fig. *P*-6, it is to be noted that the line *DB* may be regarded as the interior support of the suspended span portion of the plate and that the load by virtue of the twisting shears travels toward it in straight lines and is distributed uniformly along it. While in the cantilever portion the reversed twisting shears which change sign at *DB* cause the load to turn a corner and then travel in straight lines toward the column capitol which is the exterior support for the cantilever portion and of the plate as a whole.

Uniform Loading modifies the stress line map just derived for concentrated central load in that the shear increases from mid span toward the column and this increasing shear causes relatively sharper bending of the cantilever portion of the plate, accompanied by a movement of the point of inflection *O* toward the column *A*. The traces of the vertical planes of pure shear *OD* and *OB* remain straight lines but *OD* and *OB* make an angle with each other at *O*; cf', be', ad', are parallel to *OB* and similarly $c'f$, $-b'e$, $-a'd$ are parallel to *OD*. With this modification the construction of the stress line map for uniform load is identically the same as that for the concentrated central load.

Relative Deflections: The relative deflections along median line *XX* and the diagonal *CD* are plotted in the deflection diagram in the lower part of Fig. *P*-6. Taking the lowest point in the plate at *C* the elevation of points on the plate from this datum elevation equal distances from *c* along diagonal and median line *XX* may be readily checked analytically by the equations of Art. 639, Natural Philosophy, by Thomson and Tait, edition of 1867.

20. The Cusp or Abrupt Change in the Direction of Curves of Greatest Displacement and Equi-Potential or Least Displacement on the surface of an elastically bent plate are the resultant effect of a change in sign of the twisting shears. Since these shears change sign at all principal planes and also where the curvature passes thru zero, these loci in Fig. *P*-6 are to be found along the diagonals, the median and column lines of the panel and along *BD* as well.

21. Analysis of Square Plates: The stress line map of the square plate supported on four sides under uniform load will be substantially similar to that of the inscribed square of the continuous plate on columns with the load at the diagonal center of the panel of which *DBC* of Fig. 6 is an octant. The uniform load furnishes nearly but not quite completely the restraint of the saddle back area at the corners. Under concentrated load the radial coordinates $C1$, $C2$ and $C3$ tend to assume equal angles under the twisting shears since there is no restraint at the boundary which would cause the angularity of these radials to intersect on the half side at equal distances. In drawing this diagram for concentrated load it will appear at once that the twisting shears do not distribute the load out to the extreme corner of the plate which for this reason tends to bend upward a fixed distance from the corner which may be determined in a manner in accord with experiment from the shear curve intersections.

22. Poisson Effect: As an elastic prism is stretched it contracts laterally or if it be compressed it expands laterally in a direction normal to the applied force. The unit change in dimension laterally divided by the unit elongation is a coefficient K called Poisson's ratio. If, however, forces are applied lengthwise and crosswise to a plate the elongation is less than that which would occur were but one of these forces acting by the product of that elongation times $(1\text{-}K)^2$ as discussed at length in Section I.

Writers on the theory of flexure of plates for the past century have erroneously attempted to apply this relation to the uniform plate bent in either dish shaped or anticlastic curvature. Since the lateral contraction which determines Poisson's ratio is determined by curves of greatest shear at 45° to the line of applied forces or major principal stress in the experimental examples of pure stress from which the coefficient K has been determined, it is obvious from the spiral nature of the twisting shears and the wide variation in the angularity from 45° to the radial principal stresses in the plate bent in double curvature that the phenomena partakes of an entirely different nature.

For example, referring to Fig. P-6, at C the spiral shears become tangent to the radial principal stresses and they diverge more and more from the radial stresses toward the outer portion of the suspended span part and in like manner in the cantilever part from A toward DB.

Again, because corresponding twisting shears are of opposite sign in the tension and compression zones the effect of these shears is to cause circumferential compression or squeeze of the plate as a whole in dish shaped curvature or a circumferential stretch in anticlastic curvature. Thus the plate in bending in double curvature presents a phenomena due to lateral action entirely different from the beam in which the top or compression face becomes convex and the tension face concave due to the lateral distortion caused by horizontal shears in vertical planes.

It follows that the formula for reduction of the stress by $(1-K)^2$ is inapplicable to the plate bent in double curvature unless the coefficient be given a value wholly different from and unrelated to that found in direct tensile or compressive tests which is a needless circumlocution.

23. Relative Magnitude of Twisting and Bending Couples: That the amount of energy expended in the circumferential squeeze or stretch of the plate bending symmetrically under an isolated load is one fourth of that which would be expended were the twist and this circumferential squeeze or stretch not present may be demonstrated from consideration of the distribution of twisting moments.

We have shown experimentally that the plate bending in symmetrical dish shaped curvature or in anticlastic curvature twisting and bending functioned equally in resisting deflection. But the twisting moment is divided between two axes whereas the bending moment is about a single axis in either direction. In other words for an isolated load a given amount of bending moment M_x about Y (in case of equal moments at right angles) there is a twisting moment N_y about Y equal to $M_x/2$ but of opposite sign, and in like manner for a given amout of bending moment M_y about X there is an amount of twisting moment—$N_x = M_y/2$ and since in torsion or twist the moment is the same about two axes at right angles it follows that the sum of the twisting moments in a horizontal plane about Z equals one fourth the total statical moment which is made up of twisting and bending.

Stated in another way, if we compare the deflection of a square plate supported along two opposite sides under a concentrated central load with (2) the deflection of the same plate under the same load when supported uniformly along the four sides, then will the deflection of (1) be halved by the twist of warping about the horizontal axes of XX and YY and again halved by the twist of warping about the ZZ axis, i.e., the central deflection of case (2) is but one fourth as great as in case (1), as shown experimentally in the test of the square celluloid plate. In the case of the same plate, UNIFORMLY LOADED, because the true bending moment about the median line axes of XX and YY is one third as great when the plate is supported as in case (2) on four sides, as when supported on two opposite sides as in case (1) and the efficiency of this moment is cut in two by the twist about the ZZ axis, the deflection is but one sixth as great for case (2) as for case (1).

24. Strength of Plates in Double Curvature: The strength of plates bending in double curvature may be figured in a manner almost identical with that of the ordinary beam in which the extreme or maximum fiber stress is determined by the formula $f = Mc/I$. The M in this case will be the true bending moment about the horizontal axis normal to the line of supports. Because the twist about the vertical or ZZ axis halves the efficiency or effectiveness of the true bending moment about horizontal axes I is to be considered in this case double the moment of inertia considered in figuring the plate as resisting by beam action.

The circumferential squeeze and stretch of the plate as a whole reduces the maximum fiber stress at mid span as above computed one fourth.

There is thus a direct analogy between the resistance of the plate and the resistance of a shaft in bending and torsion. In the torsion of a shaft a twisting moment is required of twice the magnitude to produce a fiber stress of a given intensity that is required in bending moment to produce an equal fiber stress in the same shaft in bending. A like difference is thus found in plate action. It requires twice as great bending moment to produce the same fiber stress where warping and twisting of the plate occurs symmetrically about its center as is the case when the plate bends in cylindrical curvature or operates as a beam.

In comparing plates supported in different ways, because the load area which is transferred directly to the support and bounded by symmetrical planes of zero shear changes form and its center of gravity changes in distance from the support, dependent on the manner and arrangement of the supports, it is necessary to figure the bending moment about horizontal axes which is less than the statical moment by the amount of the twist about the same axis. Using this moment as M and doubling the moment of inertia of the plate along the axis considered when the plate is symmetrical the beam formula becomes applicable with and added allowance for squeeze and sketch of the plate as a whole.

In the case of the rectangular plate, because the lines of principal stress become oval or elliptic, the greatest fiber stress may be best computed by considering an equivalent square panel such that its side equals $(a^2 + b^2)/(a + b)$ in which a and b are respectively the long and short sides of the rectangle.

Where the plate is continuous, the kind and character of the continuity must be taken into consideration in a manner which will be considered more fully in the following chapter.

25. Summary: In the analysis of plate action presented in the foregoing chapter, the writer has applied the principles taught sixty years ago in Natural Philosophy by Thomson & Tait. The method of determining the true bending moment by deducting the amount of the twisting about horizontal axes is there correctly set forth tho for some inexplicable reason it has been erroneously disregarded by the engineering profession and writers on the resistance of materials. Separation and composition of curvatures into or from spherical and anticlastic components is here followed and expanded to the determination of the nature and character of the twisting shears and their delineation by stress line maps covering and explaining the stress distribution thruout the plate, the locus of the curves of displacement and equi-potential, and the squeeze and stretch of such plates when bending in dish shaped or reversed curvature. The determination of the true position of the neutral plane, and also the stress distribution on vertical planes of principal curvature follows the methods adopted in analyzing simple and continuous beams of Section II.

The relation of contours to principal stresses has been here determined for the first time, i. e., that the contour in the stress line map is coincident with the direction of the principal stress lines on middle plane of the plate and makes a slight angle in plan with the principal stresses on the surface of the place which angularity is of different sign in the tension and compression zone. These contours become tangent or coincident with the principal stress on the top and bottom surface as they cross the principal planes of the plate.

It is thought that the graphic disclosure of the state of stress at all points of the plate will enable the student to acquire a clear idea of the phenomena with which he is dealing in the study of plate action and indicate also the great economy that may be achieved by taking advantage of the principles which govern its operation.

CHAPTER XV

ANALYSIS OF COMPOSITE PLATES AND THE THEORY OF CONTINUITY WITH VARIABLE MOMENT OF INERTIA

1. Introductory: While a correct theory of plate action is of practical value in the scientific design of cylinder heads, crown sheets, plating for bins, bulkheads, and the like of homogeneous material, its most important adaptation lies in the economic design of flat slab floors of reinforced concrete in mill and warehouse construction for heavy loads and in reinforced concrete dams and bridge floorings.

The flat underside simplifies construction, reducing greatly the cost of form work, and with certain specific arrangements of reinforcing elements economy of material over beam construction for certain limits of span length results.

The early art presents a number of attempts to utilize flat slabs of concrete supported directly by columns, but with such lack of economy that Captain John S. Sewell, in a paper read before the American Society of Civil Engineers in February, 1906, asserted that,

"No extensive system of reinforced concrete floors can be economically designed without rolled steel beams or concrete ribs forming together with a portion of the floor slab, in each case what is practically a T beam."

Among twenty prominent engineers taking part in the discussion the author alone took exception to this statement outlining a new arrangement of metal by which flat plate floors might be designed more economically than by use of structural or concrete ribs.

The theory of the T beam is that in the resisting couple the steel furnishes the tensile element and the concrete the compressive resistance. With such a mode of resistance, Captain Sewell reasoned correctly that any feasible depth of the flat slab would be insufficient to permit the slab to compete economically with the beam.

Obviously some difference in distribution of metal in the slab accounts for later day success in contrast to the prior art failure.

2. Localization of Reinforcement Changes the Manner of the Storage of Internal Work. Where the reinforcement is localized in strips of relatively great ridigity, the contours of the surface which were circular in the evenly reinforced slab, Fig. 95, are bent inwardly making them convex toward the center across the strip instead of concave, as shown in Fig. 94. This produces the change from the characteristic contours of the circular plate with uniform reinforcement to that just described and causes an increase in the actual deflection of the plate as a whole because the storage of internal work is no longer divided between elastic resistance to radial and circular stresses.

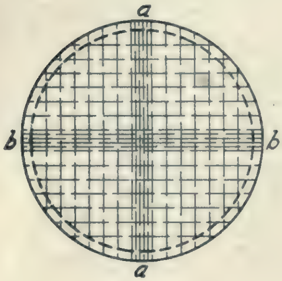


Fig. 93

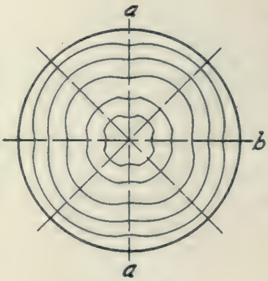


Fig. 94

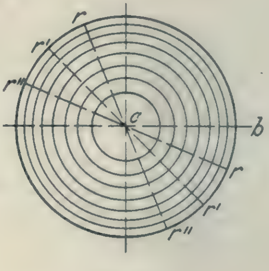


Fig. 95

Under load scallop shaped curvature occurs across the strips heavily reinforced as in Fig. 93. A state of synclastic stress such as is found in a uniformly reinforced slab with contours as shown in Fig. 95, is changed to an anticlastic state of stress with contours as shown in Fig. 94, which are alternately concave and convex toward the center of the plate. Now plate action depends on the storage of energy circumferentially by ring compressions on the upper and ring tensions on the lower surface of this plate, but when the dish shaped curvature of this plate is changed to a scallop curvature, tension and compression alternate circumferentially both above and below as the contours change from convex to concave toward the center. At the points where this change from tension to compression occurs there is no ring stress at all, consequently there can be no storage of energy in a circumferential direction to oppose deflections. Accordingly the mode of operation is reduced to that of radial resistance alone and the strength of the plate degenerates to that of mere beam strip action radially with correspondingly increased deflection and decreased ultimate strength.

Where the concentration of the reinforcement is less pronounced that is where there is but a small difference between the cross section of the metal in the strips and that of the average distribution between them such for instance as would be produced by the use of $\frac{1}{2}''$ rods in place of $\frac{3}{4}''$ in the strips and $7/16''$ rods distributed through the slab instead of $5/16''$, the change in contours would be less marked. Such moderate concentration would reduce the amount of ring compression normal to the strips of heavy reinforcement without developing the scallop shaped curvature sufficiently to make the circumferential stress change sign. The efficiency of circumferential action although impaired would not be entirely destroyed thereby until the sign of the resultant stress is changed.

3. Relation of Cross Section of Reinforcement That Resists Bending in Dish Shaped Curvature to the Form of the Contour. When the form of the contours indicates the presence of anticlastic curvature in circumferential directions which destroys the cooperation of circular and radial stresses it is important to determine the relative cross sections of the reinforcement which produces such anticlastic curvature.

If in a circular section described around the center of dish shaped curvature we plot to an appropriate scale the relative cross sections of the material upon different radii, by laying these points off consecutively in plan from the circumference of the circle toward the center and connect the points by curved lines, a figure is formed which is similar to the curve of the contour.

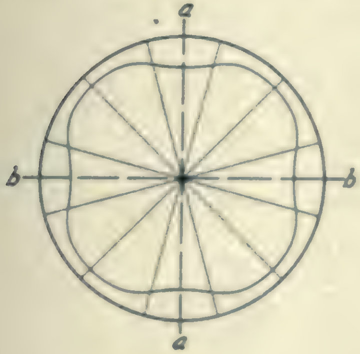


Fig. 96

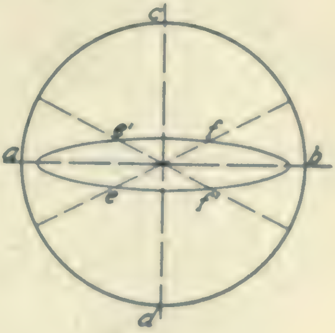


Fig. 97

Fig. 96 is a resistance diagram of the material corresponding to a non-uniform distribution of reinforcement in the circular plate supported at its edges as shown in Fig. 93. Fig. 97 represents the resistance diagram lengthwise and crosswise of a sheet of expanded metal. There is uniformity of strength in the diagonal direction *ef* and *e'f'*, but this uniformity in two directions does not control the resistance in all directions since any elongation along *e* toward *f'* would allow deflection along the axis *ab* substantially the same as though the resistance were that of beam action only. Consequently, a slab reinforced with expanded metal in one direction only cannot resist bending both radially and circularly but its operation is confined practically to that of beam resistance.

From the foregoing it appears that *plate action depends for its utility on substantial uniformity of resistance radially and circularly.* We cannot reduce the resistance in one direction greatly below that in another without changing the mode of operation in such wise that it degenerates into substantially that of mere beam resistance.

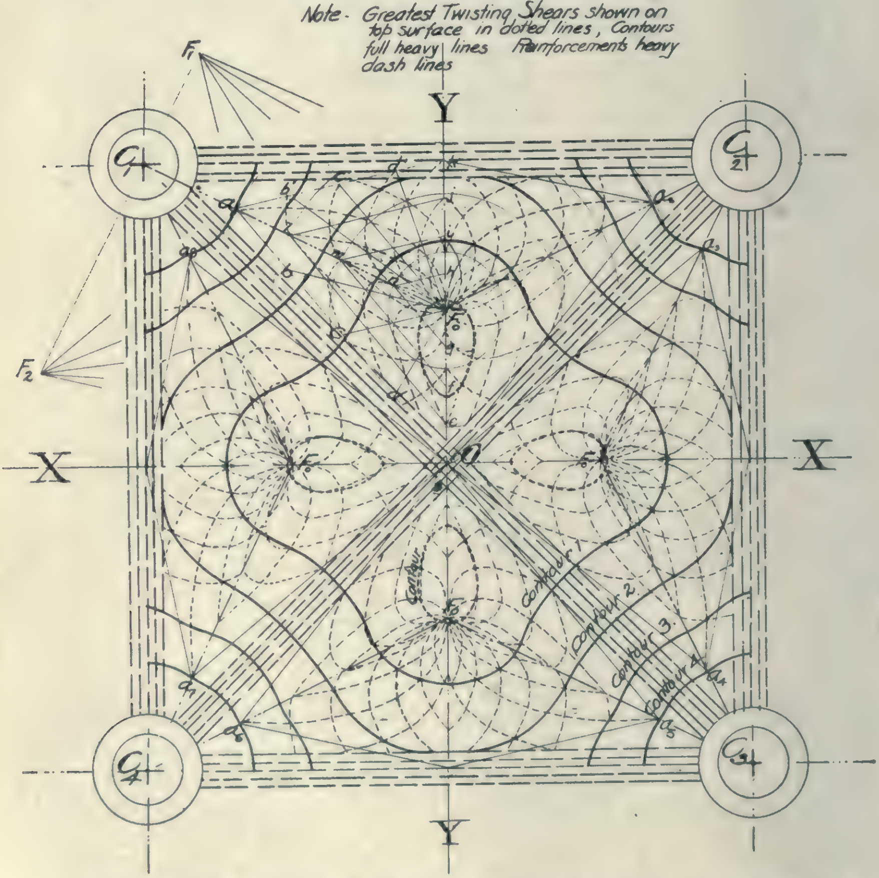


Fig. $P6^1$, Stress Line Analysis of Flat Concrete Plate Non Uniform Distribution of Steel.

Note - Greatest Twisting Shears shown on top surface in dotted lines, Contours full heavy lines Reinforcements heavy dash lines

A stress line map of a column supported flat slab reinforced lengthwise, crosswise and diagonally by strips of heavy rods over and under each other and over the tops of the posts is illustrated in Fig. $P6^1$.

Because the bands of steel rods are narrow there is a localization of rigidity such that the deflection is less at the band than at one side on a section normal to its direction.

Accordingly as compared to a uniform plate saddle shaped or anticlastic curvature occurs along the diagonals in addition to that along column lines.

If the cross section of bands are the same then the plane of division of the load between the diagonal and direct band would lie substantially in the position of $C_1 F_0 C_2$ for the upper quadrant of the plate and in a symmetrical position for other quadrants. The point of inflection, T, along $C_1 F_0$ would move toward C_1 as the relative rigidity of the plate decreases with respect to the band or toward F as it increases. The coordinates of the twisting shears may be drawn about F_0 as a focus and the contours determined as in the analysis of the uniform plate, Fig. $P5$.

The scallop shaped contours show tension in the top surface across the diagonals of the panel where in $P5$ the stress was compression in the central part of the panel.

Alternation of tension and compression occurs along the contour and therefore there can be no effective storage of energy by circular resistance. Again the curves of twisting shears present an entirely different configuration such that the shift of shear is along the diagonal toward the columns instead of normal thereto.

The resulting effect is a slight increase in strength of ten or fifteen per cent over beam action contrasted to a five fold increase brought about by uniform distribution of the same weight of metal. The difference in stiffness is far greater depending on the relative thickness to span length, it ranges from fifteen to twenty fold evincing the most radical difference in mode of operation imaginable.

4. Mechanics of Embedment. The mechanics of reinforced concrete under flexure may be summarized as follows:

The co-operation or combined action of the two materials, concrete and steel, to resist bending, depends solely on the bond between the two, which has been discussed briefly in a preceding article.

In the case of plain rods, this bond is in reality a shrinkage grip which prevents the steel from sliding thru the hardened matrix in which it is embedded, and the resistance afforded by this bond is subject to well defined laws which may be stated as follows:

The bond shear is zero wherever the tension in the steel is constant. It passes thru zero where the moment passes thru a maximum or minimum. It must be depended upon whether the reinforcement is in one direction only as in a beam, or in multiple directions as in the slab.

Bond shear generates stresses emanating from the surface of the bars which may be treated as lines of force. These lines of force follow the general laws of distribution of force thru any medium, that is, their intensity is inversely as the square of the distance from the surface of the steel on which they are generated.

These general laws enable us to investigate or follow out the part played by bond shear in the mechanics of a slab or beam. In the case of a simple beam in accordance with the law stated, the intensity of the bond shear for uniform load is zero at the center and greatest toward the end of the beam, and it is to the bond shear or the lines of force generated thereby that we may attribute the difference in the failure of an over and an under-reinforced beam.

In the case of the beam with light reinforcement, failure takes place at the center by the yielding of the steel. With heavier reinforcement, on the contrary, failure is more liable to occur toward the end by indirect tension induced by the bond shear which is greatest toward the end of the beam and which may be resisted only by the direct tensile strength of the concrete itself.

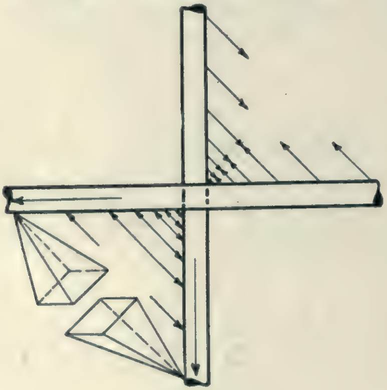


Fig. 83.

With the slab reinforced in two directions, however, the phenomenon differs from that observed in the beam. Take for example the case of the slab reinforced in two directions, bent in such a manner that the rods in both directions are brought into tension at the same time. The indirect stresses generated by the two sets of rods will under this condition react upon each other. Since the lines of force diverge from each rod they may meet and coact thru the concrete as a medium of transmission of the stress, which is not possible in the beam with one way reinforcement, since in the beam these stresses cannot coact with each other in like manner and they are of one kind only and not two kinds acting in different planes as in slab action. This fundamental difference of the stress induced by the bond shear in a beam from that induced in a slab renders the two types of structure mechanically different and necessitates their treatment in a manner which takes into consideration the difference in the operation of the indirect stresses referred to.

Another difference lies in the presence of twisting resistance in conjunction with bending resistance which results in a mode of operation different from that in beams in that twisting is resisted by shearing stresses in the slab laterally, while bending is resisted by tensile and compressive stresses in the horizontal fibers of the slab acting as a couple about the neutral axis of the vertical section. The concrete needs no reinforcement to resist twisting about a horizontal axis because such twisting is the by-product of the interference of bending moments which are of the opposite sign and of greater magnitude than the twisting moment so that the twisting moment about a horizontal axis merely reduces the amount of bending about that axis and reduces thereby the amount of reinforcement which would otherwise be needed. It is thus seen that the concrete in the slab performs a function in doubl bending wholly different from that in beam action.

5. Resistance to Deformation Furnished by the Bond Stresses of Crossed Rods in Slabs. Where two series of uniformly spaced crossed rods are embedded in a concrete slab of uniform thickness bent in dish shaped curvature so that both series of rods are under nearly equal stresses they coact with each other through the matrix horizontally in the manner illustrated in Fig. 83. In such lateral coaction the tensions and compressions produced in the concrete by the horizontal shearing forces along any rod are divided between those that are held in equilibrium laterally by similar indirect stresses generated by bond shear along horizontal rods normal to the rod under consideration and those that are held in equilibrium by the horizontal compressions in the compression zone above the tension zone which the rod strengthens.

Those portions of the indirect stresses due to bond shear which act between the rods alone and are held in equilibrium laterally constitute a series of localized self-contained systems. Linked together they constitute an essential part of the resistance to the applied circumferential stress. Their action is that of connecting the rods in such wise as to enable them to do their part in resisting circumferential tensions, and such circumferential tensions in simple synclastic curvature operate in a manner independent of the deflections and hence are independent of the propagation of radial stresses from the center of the panel outward in the suspended span or toward the central support in the cantilever portion of the slab. Those portions, therefore, of the horizontal shears that coact with each other laterally in the tension zone are independent of the compression zone radially and consequently play no part in propagating the tensions along the length of the bars radially in the manner of beam action. The separation of this portion of the shear from that which propagates such tensions will now be investigated.

The intensity of force emanating from a point in an elastic body at a given distance from that point, varies inversely* as the square of the distance from the point whence the force emanates. When the force emanates from a series of points on a line the intensity of the resultant force upon a plane at different perpendicular distances from the line is inversely as the distance from the line because of the crossing or superposition of the lines of force emanating from the several points along the length of the line.

A bar corresponds measurably to a line and although it has a certain size diametrally the deviation from the law of force emanating from a line will not be material for the purpose of our comparison of the distribution of bond stress as between that portion held in equilibrium laterally and that portion which is held in equilibrium by coaction with the compression zone.

Laterally the stresses emanating from corresponding points in two intersecting rods coact with each other and meet on a plane bisecting the angle between the rods. Because the length of the path is twice the distance to the bisecting plane the relative rigidity under the action of these stresses is directly as the distance apart of the bars so long as E_c remains constant and the material is not stressed beyond the limits of elasticity.

The relative rigidities of the paths determines the relative amount of those stresses that coact between intersecting rods, as compared with the amount that coacts with the compression concrete above. The distribution of the latter is shown in Fig. 82 in which tensions and compressions act at 45° to the rod and are of practically uniform intensity for such a short distance along the bar as the length of a mesh. This distribution is quite different from the variable distribution in a horizontal direction and the separation in the horizontal direction of the tensions on one side of the intersection or crossing of the bars and the compressions on the other.

---

*This is also the law of distribution of the force of gravity and of the intensity of light, etc.

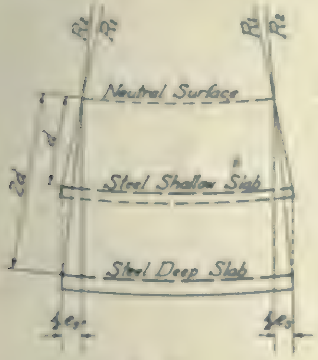


Fig 86

The rigidity of the path by which the indirect stresses travel in a vertical direction must be investigated in order to compare the relative rigidity of the horizontal and vertical paths of distribution. It will be noted in Fig. 86 in a beam in which the depth of the steel is d below the neutral plane that for a given elongation e_s in the steel the horizontal shearing deformation due to the stretching of the tension zone is twice as great per unit of depth as would occur in a beam in which the distance from the neutral plane to the steel is $2d$, and the elongation of the steel in tension is also e_s. Therefore, the rigidity of the vertical path traversed by the indirect stresses which produce these shear deformations is inversely proportional to the distance from the steel to the neutral plane and under the principle of rigidities the distribution of indirect stress horizontally and vertically will be proportional to the relative rigidities of the respective horizontal and vertical paths of distribution.

From the foregoing principles the distribution of bond stress vertically and laterally may be apportioned inversely as the distance from the steel to the neutral plane as compared with half the diagonal of the rectangular spacing of the bars. This may be taken to hold true in case the spacing of the bars is equal to, or greater than 1.4 times the distance from the steel to the neutral plane.

We may now estimate as nearly as we can the relative distribution of bond stress laterally and vertically upon this basis in the case of a slab in which the distance from the center of the steel to the top of the slab is 8″ and the bars are 7/16″ round, spaced 7″ at right angles to each other in each direction. In such a slab by referring to the j and k curves of Sec. II, it will be seen that the neutral axis would be 5″ above the steel. The distance diagonally across the mesh between the mid points of adjacent sides of this 7″ mesh is also practically 5″ so that the rigidities of the vertical and horizontal paths may be taken to be equal. Therefore, the distribution under this law would be one half in the lateral direction and one half in the vertical direction.

We will compare this distribution with the result which would be brought about by disposing the same weight of steel in the form of $\frac{5}{8}''$ rods instead of $7/16''$ and spacing the rods twice as far apart. The increase in the diameter of the bars reduces the distance from the center of the steel to the neutral plane to approximately 4 and $\frac{5}{8}$ inches and since the bars are twice as far apart the ratio is nearly as 1 to2. Therefore 2 times as great an amount of bond stress operates in the vertical direction as in the lateral and the relative stress per square inch on the bars would be as $\frac{2}{3}$ to $\frac{1}{2}$ or approximately $1\frac{1}{3}$ times as great.

The assumption upon which this relative stress is computed is based upon a radial section bisecting the angle between the two series of crossed rods. At other points around the circumference dependent on the angular direction of the crossed rods relative to the supports the distribution of bond stress in a vertical direction will be greater and that in the lateral direction less and *vice versa* with further angular change making an average approximately that which we have computed above. Thus the mat of crossed rods wherein the rods are normal to the sides of the panel are equivalent in their resistance to radial and circumferential stresses to that of a mat in which the directions of the crossed rods are parallel to the diagonals of the panels.

It may be observed that bond stresses in a mesh length are distributed with approximate uniformity along the length of the bar in their coaction with the compressions above the neutral plane vertically with lines of indirect compression and tension crossing each other at 45 degrees in that region, whereas, in a horizontal direction indirect tensions and compressions are separated from each other, operating on opposite sides of the bar in such wise that as the compressions decrease on one side of the bar, the indirect tensions increase on the other side, so that the resultant effect is a relative uniformity of distribution of all the bond stresses coacting laterally in a mesh length very like that found in the vertical direction.

It might be thought because of the increase in the intensity of the tensions toward the apex of the mesh in one direction on one side of the bar and of the compressions in the other direction on the other side of the bar that the elasticity of the concrete might be exceeded at and near the apex or points of contact of the bars. At this point, however, the bars are not opposite each other but are disposed vertically one above the other. Consequently, the bond stresses at and in the immediate vicinity of a point of contact, act predominantly in a vertical rather than in a lateral direction, so that their lateral intensity is limited to an amount within the elasticity of the concrete when subjected to reasonable test loads.

We will now compare the case of the lateral and vertical coaction of the rods where the spacing is decreased from 7″ to half that amount, or to 3½″ center to center in each direction, the rods still remaining of the same total cross sectional area. With this spacing the vertical distance from the steel to the neutral plane would remain about as before, 5″, while from rod to rod the total diagonal of the mesh would be nearly 5″. Or the half diagonal is half as great as the vertical distance from the steel to the neutral plane, as shown in Fig. 87. Because of this close spacing of the rods there is the super-position of indirect stresses vertically shown in Fig. 87 while no such super-position can occur laterally. Therefore, when the spacing is made half as great there would be a super-position of the bond stresses vertically upon the neutral plane and this path would then have become twice as efficient. Hence in light work, there is little to be gained by spacing the reinforcing elements closer than the distance from the neutral plane to the steel. Where, however, the slab is heavy, it is better not to exceed six or eight inch spacing since in the heavy slab larger rods are required and greater intensity of bond stress is developed and good design requires that

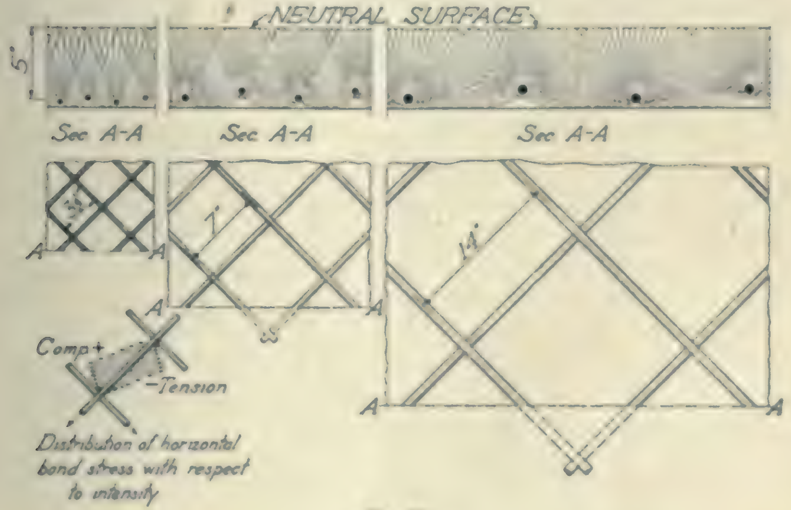


Fig. 87

the intensity of the bond stresses be kept to a minimum by more uniform distribution thereof.

6. Analytical Investigation of Bending Moments in a Square Panel. Let $OADB$ in Fig. 121 represent one quarter of a square panel of a continuous flat plate or slab carrying a uniform load of w pounds per square unit of area, in which O is at the middle of the panel, D is at the center of one of the corner supports, and L is the distance between column centers. Let $W=wL^2$ be the total panel load. Then taking the origin of coordinates at D let $x=DF$ and $y=DE$ be the coordinates of any point P on the diagonal DO. By reason of symmetry the vertical shear is zero at every point of the diagonal DO and at every point of the four sides of the quarter panel $OADB$. Also $OA=OB=\frac{1}{2}L$.

Disregarding the small part of the load that rests directly upon the column cap, a total vertical shear $=\frac{1}{4}W$ acts in that quarter part of the perimeter of the column cap which lies inside the quarter panel. But in any case the total vertical shear acting at the sides PE and PF of the square area $PEDF$ is the difference between the vertical reaction $\frac{1}{4}W$ at the support D and the load wx^2 on the area $PEDF=x^2$; hence the expression for the total vertical shear at the two edges PF and PE of any area $PEDF$ is

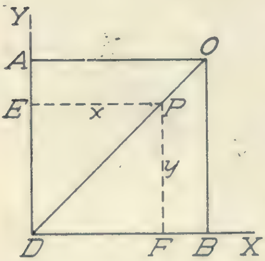


Fig. 121

$$\tfrac{1}{4}W-wx^2=S_x+S_y\text{-----------------}(1)$$

in case S_x and S_y represent components of the total vertical force perpendicular to x and y respectively which are transferred by shearing stresses across the perimeter of the area $PEDF$ within a quarter panel area.

This equation (1) is a particular case of the general condition of equilibrium for the vertical forces acting upon any given piece of the plate, since it requires that the algebraic sum of all the applied vertical forces, such as reactions of supports and loads which are applied to it inside of its perimeter, shall be equal and opposite to the sum of all the separate components of the vertical shears applied at the entire perimeter of the piece. Now, whatever size the cap may have, $\frac{1}{4}W$, and wx^2 are the only vertical forces applied inside the permieter of $PEDF$ while S_x and S_y are the only separate components of the vertical shears at the perimeter. Since as already stated *DO*, *DA* and *DB* are sections of zero shears, S_x and S_y are shears on *PF* and *PE* respectively. We have thus by symmetry

$$S_x = S_y = \tfrac{1}{2}(\tfrac{1}{4}W - wx^2) = \tfrac{1}{2}(\tfrac{1}{4}W - wy^2) \qquad (2)$$

Again, let $\frac{1}{2}M_x$ and $\frac{1}{2}M_y$ be the x and y components distributed across the width of one quarter of the panel of the bending moments due respectively to S_x and S_y. Since S_x and S_y act separately upon sections taken perpendicular to each other they each generate bending moments separately. Hence

$$\tfrac{1}{2}dM_x = S_x dx, \text{ and } \tfrac{1}{2}dM_y = S_y dy \qquad (3$$

in which M_x and M_y refer to moments across the entire panel width.

The fundamental equations (3), since they refer to applied moments as well as shears at right angles to each other treat of shears as well as bending moments that act wholly independently of each other on *OADB*, one along x and the other along y, just as truly as if they acted on two separate plates.

By symmetry $M_x = M_y$

By (2) and (3), we have

$$\tfrac{1}{2}dM_x = \tfrac{1}{8}(W - 4wx^2)dx \qquad (4)$$

or

$$\tfrac{1}{2}M_x = \tfrac{1}{8}(Wx - 4wx^3/3)$$

when taken between suitable limits.

Integrating (4) between $x = 0$ (for a mere point of support) and $x = \frac{1}{2}L$ at mid span, we have for the quarter panel

$$\tfrac{1}{2}M_x = \frac{1}{16}(WL - \tfrac{1}{3}wL^3) = \frac{1}{24}WL \qquad (5)$$

Hence, the total increase of the bending moment between a median line of a panel and a parallel edge for the entire panel width is

$$M_x = \frac{1}{12}WL \qquad (6)$$

It can be shown in like manner that in case the panel is not square, but has a length L_x and a width L_y between column centers, that the total bending moments produced between the median lines and the edges are respectively

$$WL_x/12, \text{ and } WL_y/12.$$

The total bending moments are reduced below this in case of larger caps, by reason of their size and the consequent reduction of the effective span. For example, if we assume a cap having an effective length of edge of $C=\frac{1}{5}L$, which is as small as is ordinarily allowable, (4) must be integrated between the limits $x=L/10$ and $x=L/2$ and we find

$$M_x=WL/17+ \text{..........................} (7)$$

so that the total bending moment in the practical case is less than

$$M_x=WL/16$$

For a cap of still larger size the moment will be smaller, and in case of an edge of effective length $C=0.225L$, such as is frequently employed, we find the bending moment

$$M_x=WL/18+$$

It is to be particularly noted with respect to the foregoing demonstration that it is wholly independent of any consideration whatever of the shape or position of lines of zero shear, and is also wholly independent of any question of twisting moments. It rests solely upon the fundamental equations (3) of the necessary relations defined by all writers to exist between bending moment and shear, viz. that the increment dM_x of the bending moment between two right sections at a distance dx apart is the elementary couple that is the product of $\pm S_x$ the shear on those sections multiplied by their arm dx.

That this is the correct meaning of Eqs (3) has been called in question by those who have not recognized the fact that one part of the statical moment of the load consists of twisting moments; but a reference to Church's Mechanics, Art. 240, Merriman's Applied Mechanics, Art. 48, Morley's Strength of Materials Art. 59, will show that the definition given above is in strict accordance with accepted principles, so that bending moment and statical moment are not necessarily the same. In fact, it has been previously shown that they differ from each other wherever twisting or warping occurs. It is to be further noted that the bending moment here found is the same as that already obtained in Chapter III and confirms its correctness. It was there shown that the sum of the bending and twisting moments which are called into play in transmitting the load upon the panel to the supports is equal to the total statical moment of the load about the supports.

The numerical coefficient of WL, or the so called moment coefficient for caps of different sizes relative to the span is tabulated in the accompanying diagram, in which ordinates represent C/L the relative size of the cap, and abscissas express calculated values of the moment coefficient which diminish with the decrease of the clear span between caps.

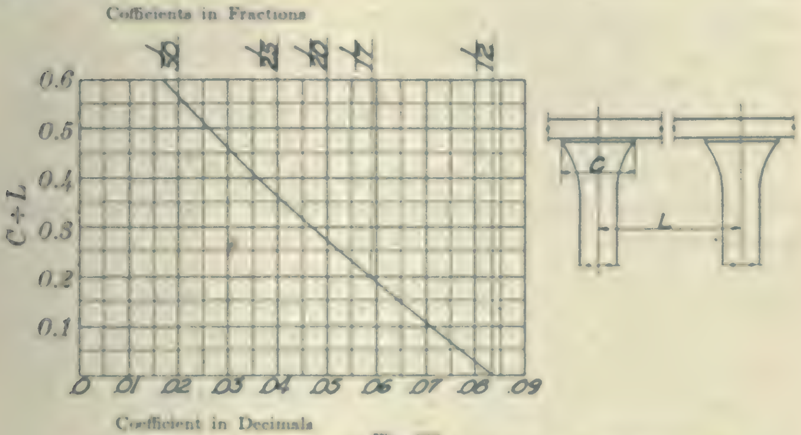


Fig. 122

Curve Showing Total Bending Moment Coefficient for Flat Slabs as Diminished by the Size of the Cap

N. B The effective dimeter of the cap is from 4 to 6 ins. less than the actual.

These coefficients apply only where there is such uniformity of distribution of metal radially and circularly, about the diagonal center of the panel and about the column centers respectively, as is necessary to make the slab imitate the action of a homogeneous plate. This uniformity is not departed from by providing a greater moment of inertia in the cantilever area corresponding to the greater moment there found than in the suspended span area, but it is departed from wherever the area of the mat is insufficient to cover the areas of synclastic curvature. Any considerable deviation from this relation so impairs the operation of the reinforcement that imitation of plate action is impossible, and the moment coefficients do not apply because of the different values of the twisting moments produced by the modification in the uniformity of the distribution of the metal.

Under the principle of rigidities, change in the resisting moment of inertia of the plate changes the distribution of moments of resistance. But where the same thickness of concrete is used throughout the slab and the flare of the capitals is depended on to give the proper shear areas under load, the only change that occurs in the moment of inertia is that brought about by distribution of the steel.

This statement is true only where the reinforcing bars are of such size that they can be compactly placed vertically in the mat over the column head. Thus, four layers of 5/16″ or 3/8″ rounds in an 8″

slab over the column cap would be within the scope of the rule laid down, but four or five layers of 5/8″ slab rods in a slab of that thickness would not come within the rule. In the latter case the moment of inertia at the support would be reduced by so large a percentage that the subdivision of resisting moment would differ greatly from that in a homogeneous plate.

The effect of variations of the moments of inertia on the distribution of resisting moments will be considered later.

7. The Magnitude of the Total Bending and Twisting Moments and the Center Deflection in a Rectangular Panel of a Uniform Continuous Slab. In order to find the total bending moment between a mid section and a margin of a panel arising from a uniformly distributed load, in a continuous homogeneous flat plate supported on rows of columns, let O be the panel center and $OBDA$ be a quarter panel of a uniform homogeneous plate supporting a load of w units per square unit of area thruout, by means of the vertical shear across HJK one fourth of the perimeter of the column cap. Assume that in such a plate the loading is carried from every point by vertical shear directly toward the column center D until it reaches the column cap, since the uniform rigidity of the plate in all directions apparently requires that the shears be so distributed as to make the loading proceed to the supports along straight lines. Then the magnitude of the total vertical shear across any section such as PQ in Fig. 123 will be the product of w by the number of square units in the area $OPQO_1$ regardless of whether the direction of the section PQ be perpendicular to OD or not. The intensity of this vertical shear will be larger the shorter the section is, provided it cuts off an equal loaded area. Hence a section in some other direction than PQ such for instance as a section perpendicular to OD will have a greater intensity of vertical shear than PQ but that fact is of no consequence in the following developments.

Take the axes of x and y along the sides OA and OB respectively and let $OA = a$, and $OB = b$. Let x and y be the coordinates of any point P upon the diagonal OD and $x + dx$, and $y + dy$ the coordinates of the consecutive point P'. Then the total vertical shear S_x across the section PF is the total loading on the area $OPFB$, hence $S_x = w(bx - \frac{1}{2}xy)$; and similarly the total vertical shear across PE is $S_y = w(ay - \frac{1}{2}xy)$, in which

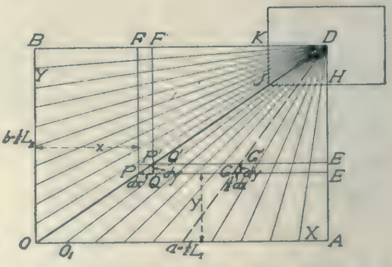


Fig. 123

$a : x :: b : y$, or $y = bx/a$

Hence, $S_x = S_y$.

It is evident from the elementary triangles with vertices at D and bases each the same fraction of the median lines OA and OB into which the loaded area may be divided as shown in the accompanying Fig. 123 when the load proceeds in straight lines toward D, that the total vertical shear S_y is uniformly distributed along PE.

The center of action of S_y is consequently at G at mid point of PE and as P moves to P' the point G moves along the line GD to G',

Let M_x denote the total difference of bending moment between median line and margin acting about the axis of y and due to $2S_x$, which moment is distributed across the entire width of the panel, *i. e.* $\frac{1}{2}M_x$ = total difference of bending moments due to S_x between OB and the opposite margin of this area of one quarter panel.

Hence $\frac{1}{2}dM_x = S_x dx = w(bx - \frac{1}{2}xy)dx$ .... (8)

For convenience of computation assume that the length of the edges of the cap have the same ratio to each other as those of the panel.

Let the size of the cap be such that $DK = a/5$, then integrating between $x = 0$ and $x = 4a/5$

$$M_x = 2w\left(bx - \frac{bx}{2a}\right)dx = bw\left(x^2 - \frac{x^3}{3a}\right)_0^{4a/5} \quad \text{..(9)}$$

Hence if $a = \frac{1}{2}L_1$, $M_x = 0.47a^2bw = 0.47\,\frac{WL_1}{8} = 0.94\,\frac{WL_1}{16}$ ... (10)

Were we to take $DK = 0.225a$ the value of M_x would be still smaller.

If the support is a mere point then $DK = 0$, and in that case $M_x = \frac{2}{3}WL_1/8$. Similar values are likewise found for M_y.

Let N_x denote the total moment about the axis of y due to S_y for the entire width of the panel, *i. e.*, $\frac{1}{2}N_x$ = the total difference of moments about y due to S_y between OA and the opposite margin of this quarter panel, hence

$$\left.\begin{aligned} &\tfrac{1}{2}dN_x = \tfrac{1}{2}S_y dx = \tfrac{1}{2}w(ay - \tfrac{1}{2}xy)dx \\ \text{or}\quad &dN_x = w(bx - \tfrac{1}{2}bx^2/a)dx \end{aligned}\right\} \text{.......(11)}$$

Hence $N_x = \frac{1}{2}M_x$, and $N_y = \frac{1}{2}M_y$

But the moment N_x is a twisting moment and not a bending moment at all, as it heretofore has been often tacitly assumed to be.

From Fig. 123 it is evident from consideration of the vertical shears upon the faces PQ and $P'Q'$ of an element that the only moment they have about y is due to displacement of the center of application on the face $P'Q'$ to the right of that on the face PQ, which is merely an element of a twisting moment and not a bending moment, which total twisting moment between PE and $P'E'$ has been calculated above about y correctly by multiplying the shear S_y upon PE and $P'E'$ by $\frac{1}{2}dx$ the distance by which the center of action of S_y has been displaced from y in passing from PE to $P'E'$.

It therefore appears by this analysis that the only shears that operate to increase $\frac{1}{2}M_x$ are those designated above by S_x, and that the shears S_y that transmit the load resting upon the area $OAHJ$ to the column and apply it as a vertical shear along HJ in doing so generate a total twisting moment $\frac{1}{2}N_x$ about the axis of y which is one half as large as the total bending moment $\frac{1}{2}M_x$ in a quarter panel between median section and margin. Now the resisting moment called into play in the slab to hold the applied bending moment M_x in equilibrium is subdivided into two parts, a resisting moment at the margin and one at median line, the relative size of these parts being controlled as is well known, by the relative rigidities of the panel at margin and mid span.

The twisting moment $\frac{1}{2}N_x$ about y which is produced by the redistribution of S_y, and the similar moment $\frac{1}{2}N_y$ about x produced by S_x is not to be confused in any way with the twisting moment independently produced by horizontal shears in vertical planes due to the kind of restraint to which the plate is subjected at the edges and to the consequent distribution of the longitudinal stresses arising from the bending moments. For example, in a continuous slab of many panels there is compression at and near the upper surface along each panel edge at mid span and tension across each edge which is replaced respectively by tension and compression at mid span at the lower surface. These states of stress are equivalent to simple horizontal shears on vertical plane sections which cut these edges obliquely but are of opposite signs above and below the neutral axis. Such opposite horizontal shears on oblique vertical planes constitute twisting moments about axes perpendicular to the oblique vertical planes mentioned. But these twisting moments due to horizontal shears are independent of those arising from vertical shears, as will be evident when we compare the case of a panel in a continuous plate such as has just been considered with that of a single isolated panel with free edges supported at the four corners either freely or built in. In this case of free edges there are longitudinal tensions and compressions along the edges as in the continuous plate but no tensions or compressions across the edge, and there are consequently no horizontal shears in vertical planes that cut the edges obliquely as in the continuous slab.

It is evident therefore that there are twisting moments due to horizontal shears in vertical planes that originate in the distribution of the longitudinal stresses arising from the distribution of M_x and M_y which distribution is controlled by the relative rigidities of the several parts of the plate and its supports, in their resistance to bending moments. These are entirely independent of the twisting moments arising from the distribution of the vertical shears which is controlled by the relative rigidities of the various parts of the plate in their resistance to vertical shears.

The radial lines of division of the load in this analysis, here taken arbitrarily by the author's co-worker Dr. Eddy may be regarded as traces in plan of vertical planes of zero intensity of all vertical shears in vertical planes whose resultant effect is the production of bending and twisting moment about the horizontal axes of XX and YY and statically correct for the purposes of determination of the total magnitude of the moments in question.

These lines of zero shear for uniform load may be utilized in the determination of the contours of curvature in the square panel without continuity or restraint at its boundary. But continuity and restraint at the boundary cause a different apportionment of the same total moment and a different position of the lines of zero shear since it introduces an interior support dividing the areas of convex and concave curvature as outlined in the treatment of the homogeneous continuous column supported plate in the previous chapter.

8. Moments at Median Section and Margins of Interior Panels in Uniform Column Supported Flat Slab Floors When Uniformly Loaded. It has been shown that the total statical moment of the load $WL/8$, acting in a half panel between mid span and supports must be reduced by some factor dependent upon the effective reduction of span due to the size of the cap in order to obtain the sum total of the applied moments that must be resisted by the slab at mid section and at one side taken together, and further that this applied moment is subdivided in its action between a bending and a twisting moment. This statement may be put in algebraic form as follows

$$m\,WL/8 = M + N \ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

in which m is the fractional multiplier by which the total statical moment of the load must be diminished on account of reduction of its effective span in order to obtain the actual applied moment $M + N$, of which the applied bending moment M is so related to the applied twisting moment N that $M = 2N$. Hence $\frac{3}{2}M = mWL/8$, or $M = \frac{2}{3}mWL/8$.

It has been further shown, that with the usually prescribed sizes of caps the value of the multiplier m is such that M is less than $WL/16$. Consequently the total applied bending moment to be resisted by the slab at mid section and edge of the cap when taken together is less than one half of the statical moment of the load between mid section and edge of the panel.

It has also been shown still further that a homongeneous plate or a floor slab that is initially flat and is reinforced in the tension zones by a fine-grained mat of crossed rods which is so firmly held together by the concrete matrix that it acts like a continuous sheet of metal when it undergoes synclastic curvature or double bending such as to produce nearly equal unit elongations e_s of the steel in two mutually rectangular directions in the dish shaped areas, requires the application of a bending moment in each of these two directions of more than twice the amount that must be applied in a single direction to produce an equal elongation e_s in that one direction alone, under cylindrical curvature. This is so because each of the moments that are mutually at right angles to each other not only produces positive bending in the direction of its action in its own plane but likewise at the same time produces negative bending in a plane at right angles thereto.

It follows that in calculating steel deformations and slab deflections due to the total applied bending moments which act to produce longitudinal stresses across vertical planes mutually at right angles to each other the deflection will not exceed one half of the deflection of the same slab in cylindrical curvature as a beam under the action of one of these moments alone, but the total resisting moment shown by the longitudinal strains may be less even than one half that in the slab when considered as a beam under the action of the total applied bending moment in one direction only, because that total applied bending moment in case the slab is thick or has a thick drop may be resisted partly by arch action, and consequently only partly by the stresses called into play by the actual flexure of the slab. The moment acting to produce flexure alone is consequently the remainder of the total applied bending moment M

Since arch action in a slab makes use of part of the compressive resistance of the concrete to resist the horizontal thrust developed by deflections in the panel while the rest of it coacts with the tensions in the steel so that the elongations e_s of the steel at any point will be those due to the applied moments diminished by the amount of the moment that expresses the arch action, as will be explained later. Hence the computed stresses in the steel will be less than those produced by the total applied bending moment, but the panel deflections in case of continuous panels since they depend mostly on compressions in the concrete, will not differ materially from the deflections of flexure for the same total applied bending moment without such reduction. For purposes of design this total applied bending moment must be subdivided between mid span and panel edge in a ratio determined either by the ratio of the moments of inertia at these sections and by restraints at the supports or by the position of the points of inflection as investigated later.

By equations (8) and (9) the differential equation of the deflections z may be written in the form

$$EI\frac{d^2z}{dx^2}=\mathrm{m}_x=\mathrm{m}_o-wb(x^2-x^3/3a)\,\ldots\ldots\ldots(13)$$

in which the origin of x and z is at the center of the panel, m_x is the value of the applied bending moment acting at any vertical section of the panel situated at a distance x from the center, and m_o is that at mid section where $x=0$.

In this equation the moment of inertia according to Section 11 Chapter VI is.

$$I=\Sigma(Aijd^2)$$

It has a value that we may designate by I_o at mid section of the panel, and another value I_x at the supports. These may be calculated as explained later in Section 6 by taking

$$A=A_1+2.5A_2$$

in which A_1 is that part of the cross section of the steel that resists by beam action, and A_2 the cross section of that part of it which is in synclastic curvature, assuming that the moment of inertia I_o is constant from mid section to the points of inflection, and I_x is constant from inflection points to supports.

In case of uniform moment of inertia such as is now under consideration with $dz/dz = O$, at $x = O$, we have

$$EI \frac{dz}{dx} = \mathrm{m}_o x - wb(x^3 - x^4 / 4a)/ 3 \ldots\ldots (14)$$

If the slab is so restrained by the caps that it is horizontal at a point at a distance $x = nL^1$ from mid span, then at that point $dz/dx = O$, and after introducing $x = nL^1$ and $W = 2wbL^1$, we obtain by (14) the moment at mid section.

$$\mathrm{m}_o = n^2(1-\tfrac{1}{2}n)\, WL_1/6 \ldots\ldots\ldots\ldots\ldots (15)$$

and by (13) we obtain the corresponding moment at the margin where $x = nL_1$

$$\mathrm{m}_x = \mathrm{m}_o - n^2\,(1 - n/6)\, wbL_1^2 = -\, n^2\,(4 - 3n)\, WL_1/12 \ldots (16)$$

From these we find the total applied bending moment between mid section and margin is

$$M_x = \mathrm{m}_o - \mathrm{m}_x = n^2\,(3-2n)\, WL_1/6 \ldots (17)$$

The fractional coefficient n here introduced to express the distance of the edge of the cap from mid span is related to C / L previously employed to express the size of the cap as follows:

In any span L we have

$$nL = \tfrac{1}{2}\,(L - C),\ \text{or}\ n = \tfrac{1}{2}\,(1 - C/L) \ldots (18)$$

so that we have corresponding value of these quantities and ratios of corresponding moments as shown in the following table:

*TABLE OF MOMENT RATIOS FOR GIVEN CAP RATIOS C/L

C/L	.0	.1	.2	.3	.4	.5
n	.5	.45	.4	.35	.3	.25
$\mathrm{m}_o/WL_1 = n^2\,(1\,\tfrac{1}{2}n)/6$	.030	.026	.021	.017	.013	.009
$-\mathrm{m}_x/WL_1 = n_2^3\,(2 - {}^3n)/6$	.053	.045	.038	.030	.022	.016
$M_x/WL_1 = n^2\,)3 - 2n)\,/6$	.083	.071	.059	.047	.035	.025
$\mathrm{m}_o/M_x = \dfrac{2-n}{6-4n}$	.375	.37	.36	.36	.35	.35
$-\mathrm{m}_x/M_x = \dfrac{4-3n}{6-3n}$	.625	.63	.64	.64	.65	.65
$-\mathrm{m}_o/\mathrm{m}_x = \dfrac{2-n}{4-3n}$	.6	.58	.57	.56	.55	.54

*This table applies when I_x the moment of inertia over the Columns equals I_o the moment of inertia ad mid span.

In this table m_o = applied bending moment at a panel mid section

m_x = applied bending moment at a panel margin

$M_x = m_o - m_x$ = the total bending moment applied between mid section and margin

The values of M_x/WL_1 in this table have been previously given in Section 2 of this Chapter. For example, in case the supports are mere points at the corners then $n = \frac{1}{2}$, and we have

$$m_o = WL_1/32, \quad m_x = -5WL_1/96, \quad \text{and } M_x = WL_1/12.$$

The results given in this table undergo modification in case the slab is not uniform by reason of variations of its moments of inertia where for instance the total moment of inertia at mid section differs from that at the margin. In cases, however, where these moments of inertia are equal it is evident from the table that the steel stresses at the margin will exceed those at mid section because $m_x > m_o$. On that account a designer would increase the marginal moment of inertia in order to make the steel stresses more nearly equal at mid span and margin and in so doing he would according to the principles of rigidities increase m_x at the expense of m_o and so decrease the ratio m_o/m_x to some extent. While the algebraic analysis of the case of unequal moments of inertia at medium section and margin meets difficulties of solution by reason of the degree of the unknown quantities involved, a satisfactory graphical analysis may be developed as follows:

9. Bending Moments and Moments of Inertia at Mid Section and Margin in Interior Panels of Continuous Column Supported Flat Slab Floors Uniformly Loaded. Let AO'' in Fig. 124 be the left hand half of a moment curve constructed as follows for a load on the span $L = 2AO$ between column centers where A is a column center, O is the mid span and the load is distributed in the same manner as it is on each of the diamond shaped areas of a slab on separated supports, viz. a load uniformly increasing from support to mid span which may be represented by the ordinates of the line AD. Subdivide the span L into some number of equal parts. In Fig. 124 $AO = \frac{1}{2}L$ is subdivided into 5 equal segments, then the ordinate of AD at the mid point of each of these segments may be taken to represent graphically the amount of that segment of the load. Lay these off successively on some convenient scale, downward from A, as the applied loads. On the same horizontal as the extremity H of the total load on the half panel assume P at a convenient pole distance from H, and connect it by a pencil of rays with the points of subdivision of the load line. Draw verticals thru the centers of gravity of the trapezoidal load areas. The moment curve may then be constructed as usual by drawing successive segments of it between these gravity verticals parallel to the successive rays of the pencil.

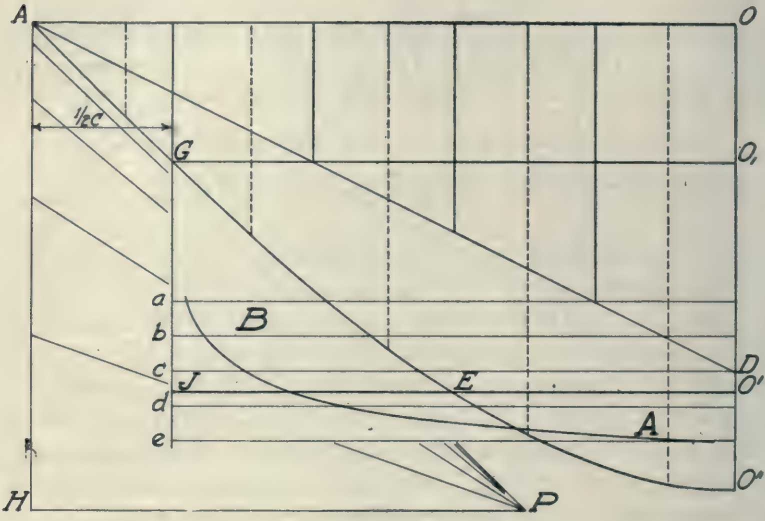


Fig. 124

The ordinates of this moment curve will represent the relative magnitudes of the bending moments produced in transmitting a load uniformly distributed upon a slab to its separated supports, because, as has been shown, these ordinates represent the only bending moments about the sides called into play by the total loading.

The side AE of this curve is straighter than the side of a parabola having an equal center ordinate OO'', and that part of the curve near the vertex O'' has a sharper curvature than such a parabola. This is due to the concentration of loading toward mid span at O. The two symmetrical halves of this moment curve do not constitute a single algebraic curve because the ordinates of the load curve does not follow a single law from column center to column center but undergoes a sudden change at O.

Let the effective span between column caps be $2GO_1 = L-C$, then GO'' is the left half of a moment curve due to that part of the load upon the effective span $L-C$, when C is the length of those portions of L occupied by the caps. In Fig. 124, $C = .2L$, and $L-C = .8L$. Assume some horizontal closing line of zero moments for this moment curve GO'' as JO'. We can assume JO' at such a level as to make the point of inflection E fall at any desired distance from mid span O'. If JO' is to be the left hand half of the actual closing line for an effective span $2GO_1 = 2JO'$ in which the center moment is $O'O''$ and the negative end moment GJ is to furnish the restraint required to keep the end horizontal, then the total bending in the span is zero. Now actual bending is proportional to total applied moment area and inversely as the stiffness or moment of inertia.

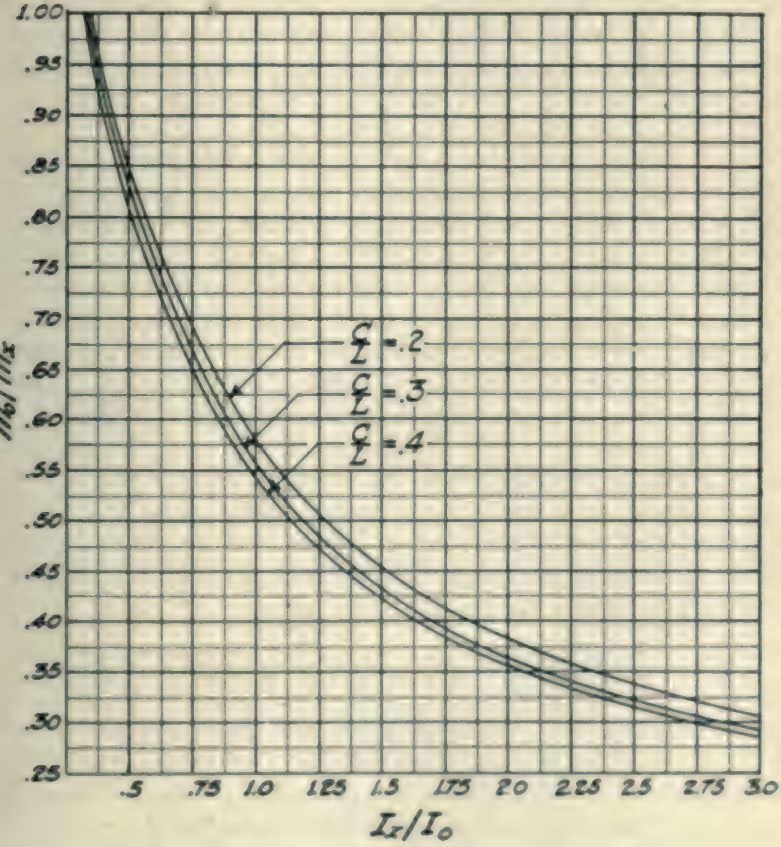


Fig. 125

Let I_o be the moment of inertia of the central part of the span between points of inflection and I_x that of the end segments. Designate the area $EO'O''$ by A, and EJG by B. Then the condition that the slab remain horizontal at the ends and at mid span is expressed by the equation

$$A/I_o = B/I_x, \text{ or } A/B = I_o/I_x \ldots\ldots\ldots\ldots (19)$$

Find the ratio A/B by planimeter measurements, and we thereby determine what ratio I_o must have to I_x in order that the point of inflection will fall on the moment curve at the point where the assumed closing line JO' intersects it. In other words, if the slab be so designed that the values of I_o and I_x have a ratio to each other of I_o/I_x equal to the measured ratio A/B, the point of inflection will be at the level of the closing line JO'. Now this required value of the ratio I_o/I_x can be secured by a suitable distribution and design of the reinforcement at the mid section and the margins of the panel.

Now assume several successive trial closing lines at nearby levels as at a, b, c, d, e for a given size of cap C/L, as for example for $C/L = .2$, and determine in each case by planimeter measurements the corresponding value of A/B. Then starting from the vertical G at the edge of the cap, lay off these values of A/B as abscissas on the horizontals thru a, b, c, d, e respectively to any convenient scale. Thru the points so found an interpolation curve may be drawn whose ordinates represent the levels of the closing lines a, b, c, d, e, etc., and whose abscissas express the corresponding values of $A/B = I_o/I_x$, and at the same time the levels a, b, c, d, e, etc., show the positions of the points of inflection where the bending moments vanish. By means of such an interpolation curve for a given value of C/L we may start with any given or assumed value of I_x/I_o and find immediately the level of the closing line, the ratio $\mathrm{m}_o/\mathrm{m}_x$, and the position of the point of inflection.

The accompanying table has been obtained by this process. It covers such a range of values of C/L and I_x/I_o as to include the usual sizes of caps and ratios of reinforcement at mid span and margin and shows moreover not only the mean position of the points of inflection that correspond to these but also the amount of the total moment in the effective span and its distribution between mid span and margin. By this table, therefore, when we have assumed any usual size of cap and any relative amounts of reinforcement at mid span and margin of the panel we are able at once to find the absolute and relative magnitudes of the bending moments at mid span and margin of the panel as well as the mean position of the points of inflection in the span.

MOMENTS AND POINTS OF INFLECTION IN DIRECT SPANS OF SQUARE PANELS

$\frac{C}{L}$	$\frac{I_x}{I_o}$	$\frac{2x_1}{L}$	$\frac{2x_1}{L-C}$	$\frac{\mathfrak{m}_o}{WL/2}$	$\frac{\mathfrak{m}_x}{WL/2}$	$\frac{\mathfrak{m}_o}{\mathfrak{m}_x}$	$\frac{\mathfrak{m}_o}{\mathfrak{m}_o+\mathfrak{m}_x}$	$\frac{\mathfrak{m}_x}{\mathfrak{m}_o+\mathfrak{m}_x}$
.2	.35	.540	.675	.0588	.0588	1.000	.500	.500
.2	.50	.516	.646	.0538	.0638	.844	.457	.543
.2	1.00	.455	.568	.0432	.0744	.580	.367	.633
.2	2.00	.385	.481	.0325	.0850	.383	.277	.723
.2	3.00	.350	.438	.0275	.0900	.306	.234	.766
.2	4.00	.328	.410	0237	.0936	.253	.202	.798
.2	5.00	.305	.381	.0212	.0964	.221	.181	.819
.3	.35	.475	.677	.0462	.0475	.975	.494	.506
.3	.50	.450	.643	.0425	.0512	.829	.453	.547
.3	1.00	.396	.565	.0335	.0603	.556	.357	.643
.3	2.00	.335	.478	.0250	.0687	.364	.267	.733
.3	3.00	.305	.435	.0212	0725	.293	.227	.773
.3	4.00	.280	.400	.0181	.0756	.240	.194	.806
.3	5.00	.270	.385	.0163	.0775	.210	.173	.827
.4	.35	.455	.675	.0350	.0362	.965	.492	.508
.4	.50	.385	.641	.0319	.0394	.809	.447	.553
.4	1.00	.340	.566	.0250	.0463	.540	.351	.649
.4	2.00	.290	.483	.0188	.0525	.357	.263	.737
.4	3.00	.265	.441	.0158	.0555	.284	.221	.779
.4	4.00	.245	.408	.0138	.0575	.237.	.193	.807
.4	5.00	.230	.383	.0119	.0594	.200	.167	.833

In this table $\mathfrak{m}_o$ is the applied bending moment at mid section, and $\mathfrak{m}_x$ that at the edge of the cap. I_o the moment of inertia of the panel at mid section and I_o that at the margin. In order to facilitate the use of the table the relation between I_x/I_o and $\mathfrak{m}_o/\mathfrak{m}_x$ is plotted in Fig. 125 which shows for the three values .2, .3 and .4 respectively of C/L how the ratio of the center moment $\mathfrak{m}_o$ to the marignal moment $\mathfrak{m}_x$ diminishes as the relative stiffness at the margin I_x/I_o increases. The striking character of the results shown in these curves makes it clear how little the ordinary assumption of distribution of the total applied bending moment into one third at mid section and two thirds at the supports corresponds to the actual distribution in reinforced slabs where I_x/I_o differs from unity, as it usually does.

10. Moment of Inertia of Reinforced Flat Plates. The moment of inertia of a plate properly reinforced with a fine-grained mat when bent in synclastic curvature differs from the moment of inertia of a beam of the same span and reinforced with the same weight of metal because under synclastic bending a given weight of steel in a slab is somewhat more effective in each of two directions. This increase of effectiveness must be taken account of in comparing the resisting moment and relative moment of inertia of the reinforcement over the supports and at mid span between the supports, and in apportioning the applied moment which will be resisted at these critical sections.

Because the greatest moment to be resisted occurs over the supports it frequently becomes necessary to obtain this increased resistance over the supports by having the slab steel spliced over the columns, and the efficiency of the laps is added to the resistance at these points. The splice or lap therefore enters into the determination of the moment of inertia over the supports and if the lap is sufficiently long it may form an effective part of the mat. The length which renders the splice efficient is determined by the question whether the lap is long enough to extend well into the rim of the area of synclastic curvature where circumferential action is largely developed. Short laps serve only to reduce the measured stress along the length of the splice while adding relatively little in proportion to their cross section to the rigidity of the mat as a whole. Hence steel laps are to be considered as fully effective in increasing the rigidity and moments of inertia of areas of synclastic curvature only when they are long enough and at the same time are correlated to other equal laps crossing them.

Supposing the size of cap and the amount and arrangement of the reinforcement given in any design for a floor, the steel in the area of synclastic curvature around the panel center may be treated for the purpose of computation as $2\frac{1}{2}$ times as efficient in every direction as a single belt in its own direction provided the reinforced area of synclastic curvature has a diameter of approximately 7/10ths of the panel width. The steel in the remaining 3/10ths of the panel width in anticlastic curvature has an efficiency the same as it would have in a beam section. This treatment is permissible only when the half diagonal of the mesh does not materially exceed the distance from the steel to the neutral plane, and the width of the crossed rod mat in the bottom of the slab is not less than 7/32nds of the sum of the sides of the panel, but 8/32nds gives a better distribution. Having thus determined the value of I_o that of I_x is to be calculated in like manner.

Where the slab is of uniform thickness and a spider of large bars is used to impart shearing rigidity to the slab at the support the neutral plane becomes a scallop shaped surface instead of a plane surface around the supports. The moment of inertia added by the large radial rods is relatively small compared with the moment of inertia of the slab bars because the latter function as a mat which the larger rods cannot effectively do. The efficiency as a tensile flange element of the radial rods of the spider is therefore relatively small and an approximate estimate of their efficiency only is sufficient.

With the values of C/L and I_o/I_x thus ascertained for the design under consideration the table in section 5 will give m_o and m_x. If the cross sections of the steel required by these bending moments at mid section and margin differ materially from those in the proposed design they should be adjusted to meet the requirements that they themselves involve.

The area of the steel section required at any section is to be found from the applied moment at that section from the equation—

$$\mathrm{m} = f_s I/id = f_s Ajd$$

as shown in Section 11 Chapter VI, in which A is the effective cross section of the steel acting as in a beam, but any steel that acts in synclastic curvature acts with a multiple effectiveness and can have its actual cross section diminished in a corresponding ratio to less than half that required in beam action. Hence the cross section

$$A = \mathrm{m}/(f_s jd)$$

is the cross section that would be required for beam action which may be written

$$A = A_1 + cA_2$$

in which A_1 is the area of that part of the steel section which acts in a beam, and A_2 the part in synclastic action which has an effectiveness c greater than 2. A_1 and A_2 need be taken only large enough to make $A = A_1 + cA_2$ satisfy the value of m given above.

The steel sections so obtained are however to be modified in turn to afford the necessary resistance whenever the size and moment of inertia of the columns cast integral with the slab is not sufficient to practically fix or restrain the slab to a horizontal position at the supports under the unbalanced loads to which it will be subjected.

At exterior panels, altho the loading may be taken as uniformly distributed the restraint at the outer edge of the floor measured by the moment of inertia of wall and wall columns is usually insufficient to hold the floor horizontal at the edge and considerations as to distribution of applied moments are to be introduced that are employed later for the case of unbalanced loads. It will be found that graphical constructions like that employed in this chapter may be used most advantageously, since they lead most directly to results of a high degree of accuracy.

11. Apportionment of Moment, Diagonal Spans of Square Panels Column supported Slabs. In considering a column supported flat plate, the structural engineer takes it for granted that the applied bending moment to which the material is subjected can be computed by ordinary beam theory, but this theory has been usually misapplied with the result that not only are the applied bending moments over-estimated as they should not be under beam theory, but the resistance of the material is also greatly under-estimated. Take the moments along diagonal lines about which the twisting moments may be disregarded since they are in equilibrium and figure these moments from the known shears that the loads produce across the diagonals by considering the center of gravity of the loads where they are known to be applied, then a rational application of beam theory to the problem will tend to make clear the fundamental errors that have been heretofore made in the determination of the magnitude of these moments.

Under uniform load, if the panel area between four posts be considered to be sub-divided by the median lines into four quarters, it is apparent that the load on each quarter is tranferred from its center of gravity at the quarter point of each diagonal to the adjacent column, and if the column support be considered to be merely a point, the total applied moment produced by the transfer is $W/4 \times L_d/4$ in which L_d is the diagonal span, or $L_d = L\sqrt{2}$ where $L =$ the direct span if the panel is square, or in any case $L_d =$ the square root of the sum of the squares of the sides of the panel whether the panel be rectangular or square.

In transferring the load to the four columns the material of the panel resists in two directions diagonally and the distribution of the quarter panel load along the diagonal which goes to each of the diagonally opposite columns is one that increases uniformly from zero at the column to a maximum at the quarter point of the diagonal and from that point decreases uniformly to zero at the diagonal center of the panel, and again increases on the other side of the center so that the moment curve of the load so distributed along the diagonal differs from the moment curve of a uniform load in that it is much steeper at the supports and much flatter throughout its center portion than the parabolic curve of uniform loading.

The effect of this distribution of load is to move the point of inflection on the diagonal toward the support as compared with its position in a continuous uniform beam under uniform load. Furthermore, it makes a cap of given diameter have an effect to reduce the total applied moment much more than would be the case were the load uniformly distributed. By reason of the fact that the load is divided in its diagonal travel in two directions the total load transferred by each diagonal is only half the total panel load, and the loads which travel in the two directions are separated by the median lines of the panel which are known axes of zero shear. The sum of the bending moments along the two diagonals considering the supports to be mere points is thus $WL_d/8$ as required by beam theory, or $WL_d/16$ in each direction. One correct application of beam theory therefore takes the diagonals of the panel as the spans for two beams that cross each other with two loads each equal to $W/4$ on each beam, the centers of gravity of which loads are situated at the quarter points of the diagonals. These four loads comprise the entire panel loading, and since they are each separately transferred to the nearest support the only bending moments they can produce on the diagonals are those just calculated.

A different application of beam theory also correct has been made previously where the entire load upon the floor was separated into lozenge shaped areas by the diagonals. These lozenge shapes areas each extended along a side of a panel, and the load upon the area produced bending moments in transmitting the load to the supports at the ends of its sides, and these were the only bending moments required to transmit the loading to the supports. The twisting moments about the axis of each lozenge are in equilibrium and need not be considered in connection with the bending moments.

These two separate analyses of bending moments along diagonals and along the sides are in no sense contradictory. They are independent and concordant. The position of the points of inflection on the panel sides for several sizes of caps and ratios of moments of inertia, etc., at mid span and supports, have been already determined graphically. The position of the points of inflection on the diagonals, etc., and the apportionment of the total bending moment between mid span and supports on the diagonals are arrived at by a precisely similar graphical process on the diagonals each under two loads of $\frac{1}{4}W$, as that previously described, while the determination of the moments of inertia at mid span and supports requires special consideration.

The moments of inertia to be employed in apportioning the total moment between the moment over the support and that at mid span will be, first I_s the moment of inertia over the column head figured as before, and second I_0 the moment of inertia at mid span which will be the sum of the moments of inertia across the panel on median lines in each direction.

This is evident since the metal about the diagonal center operates as a sheet and a sheet resists stresses in two directions at right angles to each other with even greater efficiency than it can resist stress in one direction. Thus the crossed rods in the mat about the diagonal center of the panel are able to perform more than double duty, if sufficiently widely distributed, while the steel in the areas of anticlastic curvature which extends directly to adjacent columns, supports none of the load on the diagonal strip considered, and consequently it acts with single efficiency on both sections just as it would in the lozenge shaped strips directly between columns.

Because the double moment of inertia at the diagonal mid span opposes the bending of the segments of the adjoining quarter panels, the point of inflection on the diagonal is pushed toward the column to a percentage of the total length of the span which is usually not far from .8 of the percentage of the span length at which the point of inflection lies on direct column lines when the direct and diagonal span is taken as center to center of columns in the two respective directions.

MOMENTS AND POINTS OF INFLECTION IN DIAGONAL SPANS OF SQUARE PANELS.

$\frac{I_x}{I_o}$	$\frac{C}{L}$	$\frac{2x_1}{L}$	$\frac{2x_1}{I-C}$	$\frac{m_x}{\frac{1}{2}WL}$	$\frac{m_x}{\frac{1}{2}WL}$	$\frac{m_o}{m_x}$	$\frac{m_o}{m_o+m_x}$	$\frac{m_x}{m_o+m_x}$
.5	.15	.580	.682	.0325	.0057	.583	.368	.632
.5	.20	.547	.685	.0275	.0492	.558	.358	.642
.5	.25	.515	.687	.0242	.0424	.548	.353	.647
.5	.30	.484	.692	.0190	.0355	.535	.349	.651
1.0	.15	.524	.617	.0243	.0630	.379	.275	.725
1.0	.20	.442	.615	.0200	.0568	.353	.261	.739
1.0	.25	.467	.622	.0170	.0485	.350	.259	.731
1.0	.30	.435	.626	.0140	.0405	.345	.257	.743
2.0	.15	.468	.551	.0170	.0712	.239	.194	.806
2.0	.20	.442	.552	.0145	.0623	.233	.189	.811
2.0	.25	.413	.550	.0120	.0535	.224	.183	.817
2.0	.30	.390	.557	.0100	.0445	.225	.184	.816
3.0	.15	.430	.506	.035	.0747	.181	.154	.846
3.0	.20	.407	.508	.0115	.0653	.176	.149	.851
3.0	.25	.386	.514	.0095	.0560	.170	.145	.855
3.0	.30	.365	.521	.0083	.0463	.178	.152	.848

In case of a rectangular panel the load distributed along the diagonal is constant for some distance each side of each quarter point and then decreases uniformly to zero at the diagonal center and at the column center.

The total applied bending moment in the diagonal span obtained by the preceding process when due regard is had to the practicable diameters of column caps is found to give the same total applied bending moment as that obtained in the lozenge shaped side spans in a square panel, and is approximately equal to a moment similarly figured in a rectangular panel when the mean span is taken as $\frac{a^2+b^2}{a+b}$, a and b being the long and short sides of the panel respectively. This determination is in accordance with the principles of plane hydraulic stress demonstrated in Chapter III, Section 10. The following table gives the coefficients locating the point of inflection, and the apportionment of the moments at mid span and over the supports for a rectangular panel in which L_1, the long side, equals 1.2 times L_2 the short side.

MOMENTS AND POINTS OF INFLECTION IN DIAGONAL SPANS OF RECTANGULAR PANELS

For $L_1/L_2 = 1.20$

$\frac{I_x}{I_o}$	$\frac{C}{L}$	$\frac{2x_1}{L}$	$\frac{2x_1}{L-C}$	$\frac{m_o}{\frac{1}{8}WL}$	$\frac{m_x}{\frac{1}{8}WL}$	$\frac{m_o}{m_x}$	$\frac{m_o}{m_o+m_x}$	$\frac{m_x}{m_o+m_x}$
.5	.15	577	.678	.0327	.0551	.592	.372	.628
.5	.20	.553	.692	.0288	.0475	.606	.378	.622
.5	.25	.519	.692	.0238	.0414	.576	.366	.634
1 0	.15	523	.617	.0243	.0635	.383	.277	.723
1.0	.20	.500	.625	.0214	.0548	.389	.280	.720
1.0	.25	.468	.624	.0176	.0477	.368	.269	.731
1.5	.15	.488	.574	.0198	.0679	.291	.226	.774
1.5	20	.466	.582	.0171	.0592	.289	.224	.783
1.5	.25	.437	.583	0142	0511	277	.217	.776
2.0	.15	.458	.538	.0173	.0704	.246	.198	.802
2.0	.20	.439	.550	.0144	0618	.233	.189	.811
2.0	.25	.413	.551	.0124	.0528	.234	.190	.812
2.5	.15	.447	.525	.0153	.0724	.211	.174	.826
2.5	.20	.423	.528	.0128	.0835	.202	.168	.828
2.5	.25	.402	.535	.0113	.0540	.208	.172	.833

L in the table is the length of the diagonal, $2x_1$ is the distance between points of inflection on the diagonal, $L-C$ is the distance along the diagonal between caps, m_o and m_x are the bending moments at mid span and supports, and I_o and I_x are the corresponding moments of inertia.

Under-estimation of the resistance of the material in synclastic curvature has led to defective designs in many cases in which no provision at all has been made for so restraining the synclastic cantilever areas to a synclastic mode of operation that any material transition from synclastic to cylindrical curvature is prevented under unbalanced load. Accordingly not a few designs that make a good showing under uniform load are lacking in strength under unbalanced loads. In the pioneer type, a heavy spider was utilized to furnish the required restraint. In the more advanced Spiral Mushroom Type diagonal strips and spirals combined with a lighter spider are the elements which operate to this end.

12. Deflections in Panel of a Continuous Column Supported Flat Slab Floor Uniformly Loaded.

Integrating equation (14) Section 4 we find the deflection z_1 between $x=0$ and the point of inflection $x=x_1$.

$$EI_o z_1 = \tfrac{1}{2}(\mathrm{m}_o x_1^2 - wbx_1^4(1 - x_1/5a)/6) \ldots\ldots (21)$$

in which z_1 = the magnitude of the deflection between the origin 0 at the panel center Fig. 123 and that at the point of inflection on the diagonal at a distance $x=x_1$, and I_o the moment of inertia is assumed to be the same at all sections between $x=0$ and $x=x_1$, as it is at $x=0$. In (21) the value of x_1 is to be taken from the table in section 5 as determined by the known value of n and I_x/I_o.

Again by equation (14) integrate between the point of inflection at $x=x_1$ and the edge of the cap at $x=nL$, then

$$\begin{aligned} EI_x(z_2 - z_1) &= \tfrac{1}{2}(\mathrm{m}_o L^2 n^2 - wbL^4 n^4\,(1 - \tfrac{1}{5}n)/6) \\ &\quad - \tfrac{1}{2}(\mathrm{m}_o x_1^2 - wbx_1^4(1 - x_1/5a)/6) \ldots\ldots\ldots\ldots (22) \end{aligned}$$

Hence $$EI_x z_2 - E(I_x - I_o)z_1 = \tfrac{1}{2}(\mathrm{m}_o L^2 n^2 - wbL^4 n^4 (1 - \tfrac{1}{5}n)/6) \ldots . (23)$$

from which z_2 can be obtained when z_1 is known from (21).
The total central deflection is

$$D = z_1 + z_2$$

13. Arrangement of Reinforcement in the Spiral Mushroom System. The original four way reinforcement, concentrates the steel in the cantilever area over the columns where it can act as a mat. The disadvantage of the arrangement consists in having four layers of reinforcement one above another in which the center of action of the metal is removed to a considerable distance from the extreme fibre or top surface of the slab. Unless standard specifications were strictly observed, which require the splicing of belts in their own plane, it sometimes happened that instead of having only four layers of rods over the supports, eight layers were cob-piled there, and the center of resistance of the steel was then removed to a correspondingly large distance from the surface with a resulting tendency to over-strain and crack the concrete in tension in addition to reducing the efficiency of the material as a tensile reinforcement by emplacement too far from the outer fibers, which excessive cob piling, usually resulted in increased thickness of the slab in order to cover the steel. The elevation of the head being determined by the bend of the column bars and the length of the column reinforcement which extends down to the lower floor, the strength is thus preserved but the contractor in order to save what material he could in the finish in order to make up for this loss in the amount of concrete in the rough slab would frequently make the finish so thin over the support that it would become cracked by shrinkage and present a bad surface. These difficulties, together with the consideration that the concentration of the metal in the cantilever area causes a like concentration of the compressions about the column head led to the improved Spiral Mushroom, illustrated in Fig. 126, in which the various stages of placing the steel are shown.

Stage one shows the frame or spider on which the flat spirals are placed, the spider keeping the spirals at the proper elevation near the top of the slab. In the second stage, series of rods of a width approximately two thirds the diameter of the spiral are placed in two directions parallel to the column lines, these rods being at the top of the slab over the spirals and dipping sharply downward to the bottom of the slab beyond the line of inflection. Then diagonal strips from mid span to mid span of the sides are placed in the bottom throughout as shown in the third stage. And finally in the fourth stage it is shown how parallel rods in two directions fill in the central area, each set of rods rising to the top of the slab over the saddle back areas and dropping to the bottom therefrom and continuing through the bottom of the slab in the central half of the plate.

It thus appears that the entire area of the slab is completely covered by parallel rods in two directions which arrangement is effected in four stages as just described in order to be able to place these rods at proper levels with respect to the rest of the reinforcement in a convenient manner.

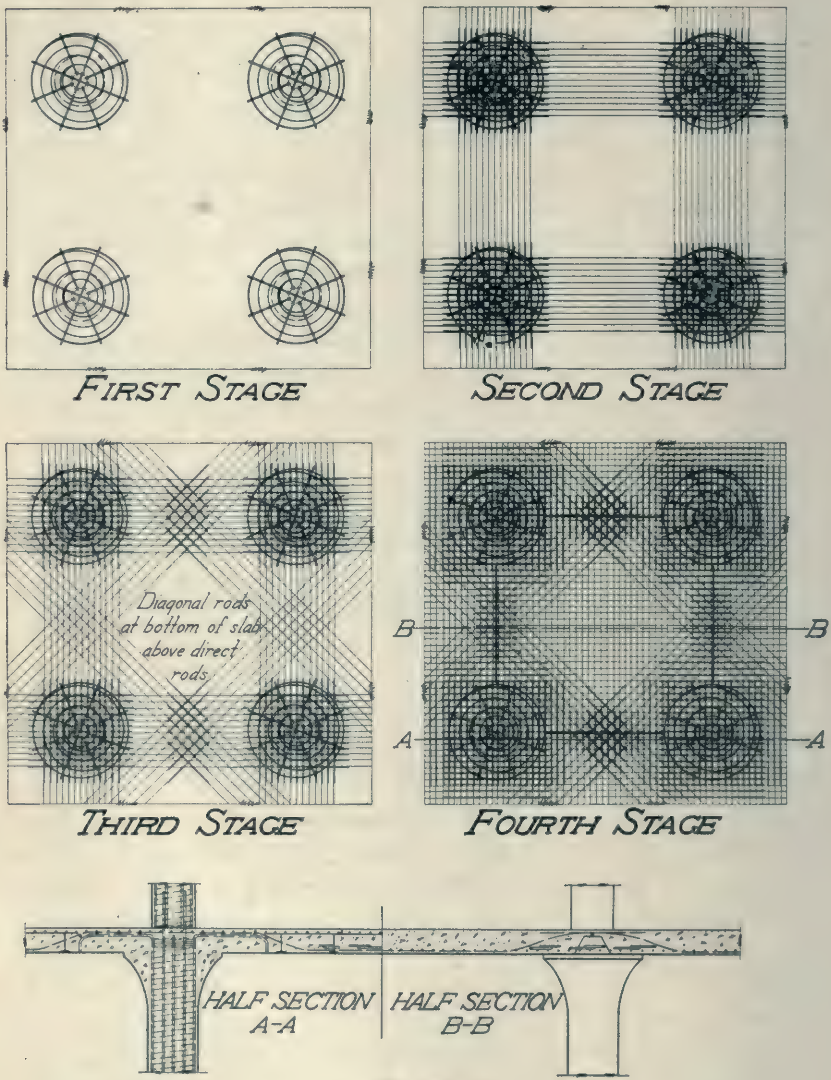


Fig. 126

The advantage of the Spiral Mushroom over the original form of construction, lies first in the reduction of the thickness of the mat of slab steel over the supports. The thickness of the mat of slab metal is reduced over supports to two layers of slab rods and a spiral rod which is generally about the size of the slab rods. There are accordingly only three thicknesses of steel in place of five layers originally which consisted of four layers of slab rods and the larger rings when the slab steel is spliced in the plane of the belts. The distribution of metal all along the saddle back areas normal to the column lines reduces the compression at the column capital, relieves the concrete of horizontal shears in vertical planes required to distribute the bending moment at the saddle back in the older type and adds considerably to the stiffness and greatly to the toughness of the slab as a continuous plate.

Comparing the distribution of metal in the Spiral Mushroom with the original, the areas of one layer of metal at mid span on the column lines in the original are eliminated by reinforcement across the panel sides. This is particularly advantageous in wall panels where a beam runs along the side of the panel, for in that position in the older construction the greater cross section of metal in two layers at mid span of the diagonal favors cylindrical curvature in a manner avoided by the localization of the maximum amount of metal at mid span across column lines in the new construction. So much synclastic curvature about the diagonal center of wall panels as is essential to imitation of plate action under unbalanced loading is hus assured.

The advantage secured by reducing the thickness of the mat over supports in the new construction is increased by the fact that the reinforcing metal between the cantilever areas normal to the saddle backs is double efficient operating on the principle of the mat as was not the case in the older type. Again the considerable vertical distance apart of the bars in the older type tended to interfere with the perfect operation of the metal as a mat. So that the advantage is secured that checking or cracking of the slab because of possible erroneous cob piling is eliminated and much greater toughness and ability to withstand over-load is secured in the later improved type brought about in part by the distribution of compressive stresses more uniformly along the bottom edges of the panels and the reduction of twisting shears distributing moments along the sides.

The oblique two way reinforcement of the Spiral Mushroom floor slabs crossing in directions parallel to the diagonals of the panels consists of belts of rods in the bottom of the slab extending from mid span to mid span of the panel sides as shown in Fig. 126, where they together with the side belts form a mat of considerable stiffness and rigidity which not only assists in bearing the tensile stresses at mid span of the sides, but by increasing the stiffness of the saddle back areas in all directions affords resistance more nearly like a unitary structure with a self-contained system of stresses, less subject to warping and bending unsymmetrically in either direction than one with the same amount of steel all placed in parallel relation. This arrangement tends to equalize the stiffness of the saddle back with the circumferential cantilever and so to tie them both together as to increase their efficiency by their coaction with each other.

Not only is far greater toughness and capacity to withstand unbalanced loading developed by the new arrangement but a most remarkable increase in rigidity under uniform loads is also secured. The diagonal bands tend to equalize the anticlastic curvature along the column and median lines and to oppose elastic resistance of the steel to the stretch of the concrete matrix as a whole thru the area of anticlastic curvature.

The two fold increase in relative stiffness of the same volume of concrete and weight of steel found in comparative tests of the spiral type with the two way construction designed according to the Chicago or New York codes is thus accounted for.

14. Computation. In considering the cross section of metal, that over the cantilever may be treated as a mat along column lines. That in the top of the slab between the cantilever edges is to be treated as developing only the equivalent of its cross sectional area in resisting the applied moment on the beam principle since there is no circumferential action possible in the saddle shaped portions of the slab.

The double moment of inertia of the entire section of steel in the bottom of the slab is to be counted just as in figuring the homogeneous plate.

In the ordinary two way construction about .6 of the section at the median lines of the panel may be doubled in figuring the effective moment of inertia and approximately .7 when the four way is that of the original mushroom type.

Circumferential squeeze in the dish shaped areas and stretch in the saddle portion can be counted on to reduce the steel stress as above figured approximately one fourth.

15. Steel Ratios and Minimum Thickness of Slab. Since slabs should be so designed that the steel would yield before the concrete, it is important to determine how large a percentage of steel may be introduced without passing this limit. This, however, is dependent upon the relative thickness of the slab. In case of the continuous slab, the ratio of the depth to side span involves somewhat different considerations from that of the ordinary beam since the minimum thickness of the slab on a diagonal span is relatively less than is permissible in beam construction. That part of the thickness of the slab which serves as fireproofing and which is constant for any given size of rods bears a greater ratio to the depth than is the case in beam construction. Furthermore, the depth necessary for proper embedment over the support and the massing of the steel at the support reduces the effective lever arm of the steel in the cantilever to j_2d, which reduces it to a relatively greater extent than at mid span where the effective depth becomes j_1d as it does in the case of one way slab construction also. This reduction in effective depth for fireproofing, embedment and massing of the steel is, however, a constant for a given size and arrangement of reinforcement and for the continuous slab it is found that we may accordingly take this constant as follows:

2 ″ for 3/8″ rod reinforcement (or smaller sizes)
2¼″ " 7/16″ " "
2½″ " 1/2″ " "
3 ″ " 7/8″ " "

The limiting minimum thickness of slab then may be stated as $L/48$ plus the constant just tabulated for the various sizes of rods.

In case square twisted rods are used, the same constant should be used for 3/8″ square twisted as for 1/2″ rounds, and for 1/2″ square twisted the same constant as for 5/8″ rounds. This will give a satisfactory minimum thickness of slab for all spans of this type, L being the longer direct distance between column centers.

For square panels supported on beams or walls at the edges, the constant for the various size of reinforcing rods should be one inch less than that tabulated above, and the minimum thickness taken as $L/48$ plus this new constant.

Where the panel is rectangular and not square, and supported on the sides by beams, the same rules will hold by using $(a^2+b^2)/(a+b)$ in place of L for the span of the square panel, in which a and b are the spans in the two directions respectively.

Determination of reinforcement required in slabs. In case of beams with different values of N the ratio of depth to span, the proper values of the steel ratio p were obtained in Chapter VI page 166. In case of slabs the ratios so obtained should not be exceeded, but in reckoning the reinforcement of slabs account must be taken of the fact that the steel is in multiple directions so that the reinforcement per belt will be only a fraction of the total permissible steel.

In the continuous slab with four way reinforcement and Mushroom heads, having given the ratio of the thickness to span taken center to center of columns, determine the percentage of steel from the diagram for beams, page 166, and divide this by 2 to find the limiting percentage of steel for each belt.

In the slab with two way reinforcement, supported on columns, the side belt may be made a little heavier than in the four way belt type, .6 of the percentage given for beams being permissible.

In slabs not reinforced with a supplemental cantilever frame, but with a depressed head instead, the minimum thickness should not be less than the minimum thickness allowed for the former type, but the steel ratio may be made .55 that for beams instead of .5 which is permissible with mushroom heads.

In square slabs supported on beams or walls the percentage of steel in the strip occupying the middle third of the panel each way should not exceed that for beams of similar proportions of depth to span. In case the panel is a rectangular panel, the percentage of steel should not exceed that for square panel having a side equal to $(a^2 + b^2)/(a + b)$ which we have previously used in determining the minimum thickness of span for this form of panel.

With the above interdependent limitations as to the minimum thickness and maximum steel ratio in mind the designer has to determine how far in any given design he shall deviate for any reason from them. By making N smaller and the slab consequently thicker he will be able to reduce the deflection to such figures as may be desired or required. But when such deviations are made the proper relationship between steel ratio and slab thickness is still to be determined from the diagram page 166 as has just been done for the case of minimum thickness.

16. Increase in Depth of Slab Required for Increase of Load. For a given span and given size of capital the moment to be resisted increases directly as the load. Now were the area of the steel increased in the same ratio as the slab thickness the resistance of the slab would be increased as the square of the depth. Therefore, from this consideration the thickness would increase in the ratio of the $\sqrt{w}$ in which w would represent the sum of the dead and live loads per unit area. On the other hand, as the thickness of the slab increases relative to a given span the percentage of steel that it is permissible and economical to use also increases as before shown. Moreover, the amount of arch action increases with the thickness of the slab relative to a given length of span, while the dead weight per square foot as well as the thickness, increases at a less rapid rate when the percentage of steel is increased, so that the sum total of these variable elements renders the increase in net thickness or depth of the slab, from the center of the steel to the center compression fiber, nearly as the cube root of w, a conclusion which is in accord with good practice and quite different from the Joint Committe rule that the depth from the center of the steel to the top of the slab should increase as the square root of w, the loading. For the same load per foot, as the span increases, the bending moment for a given ratio of the diameter of cap to the span length increases as the cube of the span, and since the resistance of the concrete and metal for a given percentage of steel increases as the square of the depth, the increase in thickness of the slab with the increase in span for the same unit load should be directly as the increase in span. The fireproofing should be the same for each.

Eddy, Trans. Am. Soc. C. E. Vol. LXXXI pp. 1173, gives the following formula and Table for d the distance from the center of the steel to the the top surface at mid span:

$$d = L\frac{4w_o + 2,800}{10^4} \text{ inches;} = \text{effective thickness}$$

L = span, in feet; w_o = live load, in pounds per square foot.

Table 3.—Effective Thickness d of Flat Slabs.

Span in feet	$w_0 = 100$	200	300	400	500	600	Differences
14	4.48	5.04	5.60	6.16	6.72	7.28	0.56
16	5.12	5.76	6.40	7.04	7.68	8.32	0.64
18	5.76	6.48	7.20	7.92	8.64	9.36	0.72
20	6.40	7.20	8.00	8.80	9.60	10.40	0.80
22	7.04	7.92	8.80	9.68	10.56	11.44	0.88
24	7.68	8.64	9.60	10.56	11.52	12.48	0.96
26	8.32	9.36	10.40	11.44	12.48	13.52	1.04
28	8.96	10.08	11.20	12.32	13.44	14.56	1.12
30	9.60	10.80	12.00	13.20	14.40	15.60	1.20
32	10.24	11.52	12.80	14.08	15.36	16.64	1.28
Differences	0.64	0.72	0.80	0.88	0.96	1.04	

This formula gives satisfactory values of *d* for the cap diameters of .2 the span length but requires modification and adjustment for other ratios of cap to span length which may be readily determined from the relative magnitude of the coefficients of the applied moments. It will be noted that these thicknesses follow very closely the above approximate relations, and to these thicknesses is to be added the necessary fireproofing.

17. Size and Spacing of Rods in Flat Slab Construction. The plate action which we have been treating, brought about by indirect stress, depends for its efficiency upon the dissemination of the steel throuhg the mass of the concrete. Large rods and wide spacing should accordingly be avoided. Where the steel percentage is very small it sometimes happens that 5/16 or 3/8 inch rods, the smallest practical sizes, will be spaced as far apart as eight inches between centers. While reasonable results will be secured with such spacing where the percentage of steel in the belt is as small as .22, where the percent of steel is larger the spacing should be closer.

One of the common errors in flat slab design is the use of such rods as 1/2 inch or even 5/8 inch from eight to twelve inches between centers with the expectation of securing results in keeping with a more scientific and uniform dissemination of metal.

Half inch rods should not be used in slabs thinner than twelve inches at the columns, nor 5/8 inch rods in slabs thinner than fifteen inches if the best results are to be secured.

18. Summary. In important respects flat plate theory diverges so widely from beam theory that the character and extent of this divergence may be profitably discussed in some detail.

Plate action requires simultaneous curvatures of orthogonal vertical sections both at panel center and supports that do not differ too widely from each other in the column supported slab, otherwise the theory here developed becomes inapplicable. A reinforcement of crossed rods running lengthwise and crosswise in a rectangular concrete slab would not operate as a mat according to plate theory if the slab be supported on opposite sides only; one set of rods only would then be under strain.

Limitation of the narrow dimension of a panel to not less than five-eighths the long dimension recognizes this principle. The combination of continuity and plate action is possible in the continuous reinforced concrete plate only when there is present a relatively uniform crossed rod mat of reinforcement both in the tension zone at the top of the slab about the columns and at the bottom of the slab about the diagonal centers of the panels. Plate action is impossible with narrow belts of rods as demonstrated by stress line analysis of such reinforcement in Fig. *P*6.

Continuity is interfered with in square or rectangular slabs supported by two-way beams of equal strength and can be depended upon but for a very small increase in stiffness and strength. On the other hand, if the support be circular and the slab fully restrained at the support, fulfillment of the same conditions as for the column supported slab, develops the full combined efficiency of these two kinds of resistance.

With uniform mats of rods, a striking similarity appears in the stress line maps of the homogeneous and reinforced concrete plates when their horizontal compression faces are compared, but in the vertical plane that wide divergence in stress distribution is found that was illustrated in the comparative analysis of homogeneous and reinforced concrete beams of Sec. II.

In the continuous column supported flat slab, interference of the detrusions of horizontal shears in horizontal planes occurs in a manner characteristic of the continuous beam, setting up dome action in the plate where arch action is caused in the beam.

Dome action in common with plate action presents circular and radial resistances combined and hence may be compared and coordinated in the same manner as arch action and simple flexural resistance were compared in Sec. II, in the treatment of continuous beams.

While such dome action is measured by from 8 to 16% of the whole resistance in the warehouse slab in a reinforced plate, for a high dam, because of the relatively greater thickness to span length for high heads, its importance may be three to four times as great, rendering the one way multiple arch structure greatly inferior in economic stability to the column supported plate.

The continuity of the column supported slab depends in part upon the stiffness of the supporting columns. The stiffness of the latter decreases with the cube of the length and increases for the same per cent of steel as the cubic of their breadth or thickness. Frequently designers put in a 16-inch column for a roof deck slab costing more for the reduced diameter of form than the extra concrete to build an eighteen inch section, with its enhanced strength and stiffness. This matter will be dealt with at length in the following chapter for which the reader is indebted to Dr. H. T. Eddy as also for his joint work on parts of the two preceding chapters.

CHAPTER XVI

UNBALANCED MOMENTS IN MONOLITHIC FLOORS AND COLUMNS

Introductory. A monolithic reinforced concrete slab floor with its column supports is an indeterminate structure, the deformations due to bending under unbalanced loads being distributed between the columns and the slab as shown in the accompanying figure.

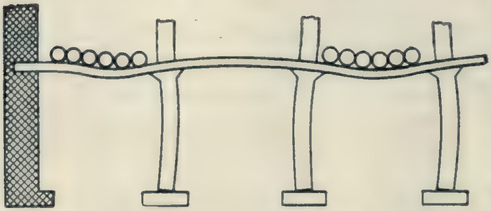


Fig. 132

Thus the panel of the slab at the left in Fig. 132 built into the wall is bent downward by the load on that panel. The wall furnishes restraint at that end of the panel as does the column above and below at the right. The column, restrained at the footing below, is bent in a reverse curve while the unloaded panel is bent upward. The next panel which is loaded is bent downward. The bending of the column in each case is toward the unbalanced load. Like bending occurs where there are beams built as a monolith with the columns in place of the slabs.

Since reinforced concrete buildings are indeterminate structures the resistance to applied moments arising from unbalanced loads must be sub-divided between the slab itself and the supporting columns according to their relative rigidities, but the equations for determining these resistances on the principle of least work are too complicated for practical use. A more simple method, that is sufficiently exact for practical purposes may be developed as follows:

The method depends in the first place upon an investigation of the properties of continuous beams which are also indeterminate structures, secondly upon the investigation of the action of unbalanced loads independently of that of the balanced loads. Since flat slab floors like other engineering structures are regarded as elastic, the stresses due to separate kinds of loading such as balanced and unbalanced loads may be computed separately, while the total actual deformations will be the algebraic sum of the component deformations arising from these separate loads.

This Chapter will treat the deformations and stresses arising from unbalanced loads alone, assuming that a separate computation of balanced loads has been made.

2. Theory of the Uniform Continuous Beam. Consider the action of unbalanced loads upon a beam *AB* Fig. 132 whose weight is to be neglected because its weight constitutes part of the balanced loads. Assume further that the Span $OA = L$ is free from unbalanced loads, but that the other spans may have unbalanced loads applied to them of arbitrary position and magnitude and such as to cause the span *OA* to fulfill the conditions that will be required of it at the supports *O* and *A*, or elsewhere. The first of the conditions that will be required is that the beam shall be bent in such a manner as to tip thru a certain assigned small angle *v* at *O*, where it is integrally joined either to a single column such as *OD* below *AB*, or to two columns *OD* and *OC*, one below and the other above *AB*.

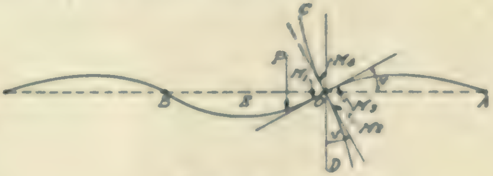


Fig. 132

In reality the foregoing requirement embraces two conditions at once, viz., one is the pinning of the beam to the support *O* regardless of its direction at that point, and the other is restraining it at the fixed slope *v* at *O*. Let a second requirement be that the beam be also pinned to the support *A* either with or without restraint as to direction, as will be stated later. All these conditions and restraints will be brought about by loads such as *P* situated in spans at the right and left of *AO* and by integral connections with columns, etc. Let the tipping of the beam at *O* thru the angle *v* be regarded as produced by the application of the moment M_1 in the beam just at the left of *O*, which at the same time will cause the column or columns integral with the beam at *O* to become inclined to the vertical at the same angle *v* by which the beam is made to incline from the horizontal at *O*.

This applied moment, M_1 may be taken as due to unbalanced loads *P* at the left of *O* and reaction at *O*. The moment M_1 is held in equilibrium by the resisting moments M_2 and M_4 in the columns below and above *O*, assisted by M_3 the resisting moment in the part of the beam immediately at the right of *O*. Hence,

$$M_1 = M_2 + M_3 + M_4 \text{ .... } (1)$$

is the equation of equilibrium of the applied and resisting moment called into play at *O* by the unbalanced load. It should be specially noted that since no tipping of the beam whatever can be produced at O by any balanced loads, v depends solely upon the unbalanced part of the total load, and will be unchanged whether a balanced load is present or absent, because in an elastic structure the independent deformations are merely superposed on each other. The same is true at every support and at mid span between supports, but at all other points the inclination will depend both on the balanced and the unbalanced loads.

Apply to the span OA the known general theorem that the sum total of the numerical values of the positive bending moment at mid span added to one half the sum of the numerical values of the negative moments at the ends is equal to the constant $\frac{1}{8}wL^2$ when w is the uniformly distributed load per unit of span. In this case $w=0$, hence

$$M_o - \tfrac{1}{2}(M_3 + M_A) = 0 \tag{2}$$

in which M_o is the positive moment at mid span, M_3 and M_A are the negative moments at O and A respectively. These moments are not due to the balanced loads in any way whatever, but are the parts of the total moments at these points due to the unbalanced loads alone. The total moments will be obtained by adding to these the moments due to the balanced loads.

In order to investigate the moments, the slope, and the deflections occurring in the span OA which are due to unbalanced loads in spans other than OA, let M be the bending moment at any cross section of the span OA situated at a distance x from O, then

$$M = EI\frac{dy^2}{dx^2} = Rx + M_3 \tag{3}$$

in which R is the reaction at O, or the shear in a cross section immediately at the right of O.

Integrating (3) we have

$$EI\frac{dy}{dx} = \tfrac{1}{2}Rx^2 + M_3x + C \tag{4}$$

but at O, $x=0$, $\frac{dy}{dx}=v$, or $C=EIv$ (5)

Substituting this in (4) and integrating, we have

$$EIy = Rx^3/6 + \tfrac{1}{2}M_3x^2 + EIvx + C' \tag{6}$$

At O, $x=0$, $y=0$, hence $C'=0$

Where $x=L$, $y=0$, hence

$$\tfrac{1}{2}RL^2 + \tfrac{3}{2}M_3L + 3EIv = 0 \tag{7}$$

These equations express relations that hold true for all degrees of restraint at A such as would be produced either by fixing the beam at A at any given inclination to OA, or by applying any assigned bending moment to the beam at A or at any other given point of OA. This is evident because with (7) given, and either the slope $\frac{dy}{dx}$ or the moment M given at some known point of the span distant x from O, the determination of M_3 in terms of the known inclination v would follow from eliminating R between (7) and either (3) or (4).

1st For example at $x=L$, let $\frac{dy}{dx}=-v$, then

by (4) $-EIv=\frac{1}{2}RL^2+M_3L+EIv$

Hence by (7) and this equation we have

$$M_3=-2EIv/L \qquad (8)$$

In this case $R=0$, and M has the constant value M_3 at all values of x and hence the beam is bent in the arc of a circle, and the sum of the bending moments at O and A is $M_3+M_4=4EIv/L$. These conditions could be brought about most readily by equal loads symmetrically situated in the spans at the right and left of the span OA.

2nd At $x=L$, let $M=0$, in which case the beam terminates at A where it is simply pinned to the support, so that it is free to assume any direction at that point compatible with the slope v at O. Hence by (3)

$$0=RL+M_3, \text{ or } R=-M_3/L$$

Substitute this in (7), hence

$$M_3=-RL=-3EIv/L \qquad (9)$$

and from (4) we find the slope at A is $-\frac{1}{2}v$.

3rd At $x=L$ let $\frac{dy}{dx}=0$, in which case the beam is built in or fixed horizontally at A. Then by (4) $0=\frac{1}{2}RL^2+M_3L+EIv$ which combined with (7) gives for this case

$$M_3=-4EIv/L \qquad (10)$$

$$R=6EIv/L^2=-\tfrac{3}{2}M_3/L$$

While by (3) the bending moment at A is

$$M_A=RL+M_3=-\tfrac{1}{2}M_3$$

In this case by (3) we find that $M=0$ at $x=-M_3/R=\frac{2}{3}L$. But by (9) and (10) M_3 is in this case the same as in a beam simply pinned at a point distant $\frac{3}{4}L$ from O, because if the length from O to A in (8) be made equal to $\frac{3}{4}L$ we get the value of M_3 given in (9).

Furthermore we find in this case by (6) that at $x=2L/3$ we have $y=2vL/27$, which is the elevation of the point where $M=0$ above the horizontal line OA. By (3) the positive bending moment at A is $M_A=2EIv/L$.

Other cases may be treated in a similar manner with results which have been given for convenience of comparison in the following table in which either the moment at A where $x=L$, or the slope at that point may be given. With either one given the other properties of this span OA of the beam become completely known as already explained, and we have values such as are given in the table.

TABLE OF END MOMENTS AND SLOPES, ETC., OF AN UNLOADED SPAN *OA* WHEN THE END SLOPE $\left(\frac{dy}{dx}\right)_{(x=o)}=v$ IS GIVEN, AND EITHER $\left(\frac{dy}{dx}\right)_{(x=L)}$ or $M_{(x=L)}$ IS ALSO GIVEN.

	$M_{(x=o)}$	$M_{(x=L)}$	$M_{(x=\frac{1}{2}L)}$	R	$\frac{dy}{dx}_{(x=L)}$	$\frac{dy}{dx}_{(M=o)}$	$X_{(m=o)}$
1	$-2EIv/L$	$-2EIv/L$	$-2EIv/L$	0		0	
2	$-3EIv/L$	0	$-3/2EIv/L$	$3EIv/L^2$	$-\frac{1}{2}v$	$-\frac{1}{2}v$	L
3	$-4EIv/L$	$+2EIv/L$	$-EIv/L$	$6EIv/L^2$	0	$-\frac{1}{3}v$	$\frac{2}{3}L$
4	$-5EIv/L$	$+4EIv/L$	$-\frac{1}{2}EIv/L$	$9EIv/L^2$	$+\frac{1}{2}v$	$-\frac{7}{8}v$	$\frac{5}{9}L$
5	$-6EIv/L$	$+6EIv/L$	0	$12EIv/L^2$	$+v$	$-\frac{1}{2}v$	$\frac{1}{2}L$

The usual conditions fulfilled by a continuous beam which has an unbalanced load on a single span only, such as OB, are represented by the values in cases 2 or 3, or some intermediate values dependent upon some degree of restraint at A between perfect freedom of motion, *i. e.* $M_A=0$, and perfect horizontal restraint, *i. e.* $\left(\frac{dy}{dx}\right)=0$ at the point where $x=L$.

In case the continuous beam is integral at A with columns, or is restrained by other parts of the structure, that will cause the beam to approach more or less closely to case 2.

The same considerations hold respecting the columns OD and OC below and above O as apply to the span OA at the right, viz.: the ends of the columns away from O will in general have a degree of restraint that is nearly perfect either by their footings or junctions with other parts of the structure, and they may be assumed without appreciable error to require the same coefficient b intermediate between —3 and —4 in expressing their resisting moments at O, as does OA, *i. e* where $x=0$.

$$M = b\,E\,I\,v/L \qquad \text{(11)}$$

is an equation which applies to either of the resisting members at O in case suitable values be ascribed to I and L in each member, while common values of b and v apply to all of them.

Now using the subscripts previously employed to distinguish the several members that meet at O, viz., 1 for the beam at the left, 3 for that at the right, 2 for the column below and 4 for that above AB, we hence have by (1) and (11)

$$\frac{M_3}{M_1} = \frac{I_3/L_3}{I_2/L_2 + I_3/L_3 + I_4/L_4} \qquad \text{(12)}$$

with similar expressions for M_2/M_1 and M_4/M_1. These expressions will show how the applied moment M_1 is subdivided among the resisting moments of the columns and span OA in case the structure is of homogeneous materials. In case of composite materials the formula will not be quite so simple because it will not be possible to cancel a common value of E from both numerator and denominator, as has been done in (12).

In case of a deck slab the column above O is absent, and in that case $I_4=0$ in (12). For any given or assumed value of the applied moment M_1 the effect of the absence of OC from the structure is to increase both M_2 and M_3, since the removal of OC causes a loss of rigidity which obliges the remaining parts to furnish the entire resistance instead of merely a fraction of it. For example, were $I_2/L_2 = I_3/L_3 = I_4/L_4$ in (12) then would $M_2 = M_3 = M_4 = \frac{1}{3}M_1$. But were $I_4=0$ and $I_2/L_2 = I_3/L_3$ then $M_2 = M_3 = \frac{1}{2}M_1$. In like manner were the beam to end at O, and not extend into the span OA, then $I_3=0$ and the entire applied moment M_1 must be resisted by the columns with which it is integral at O, so that M_2 and M_4 would be much increased thereby.

The case that was considered in what has preceded is one in which the same degree of restraint is assumed to exist in both the columns and in the beam at the extremeties designated by A. In case, however, where the beam terminates at A and merely rests upon a support at that point while the ends of the columns distant from O are built in so as to remain nearly fixed in direction it is evident from comparison of (9) and (10) that we must replace I_3 in (12) and the similar equations by $4I_3/3$, a reduction of rigidity of OA that will make M_3 less, and both M_2 and M_4 larger, than in the previous case.

3. The Moments and Deflections in a Single Uniformly Loaded Span of a Continuous Beam. Let the span OB be assumed to be of length equal to $OA = L$. Let it be uniformly loaded with an unbalanced load of w per linear unit, so that it constitutes the middle one of three equal spans on four symmetrical supports, and otherwise the continuous beam is supposed to be symmetrical in all particulars to the right and left of the center E of the span OB, there being no loads upon the end spans.

The differential equation of the neutral axis of OB in case the origin is taken at O and x is measured to the left is

$$EI \frac{d^2y}{dx^2} = M_1 + \tfrac{1}{2} w L x - \tfrac{1}{2} w x^2$$

in which M_1 as previously is the bending moment just at the left of O. Integrating we have

$$EI \frac{dy}{dx} = M_1 x + \tfrac{1}{4} wL x^2 - wx^3/6 + C$$

At O, $x = 0$, and $\frac{dy}{dx} = v$, hence $C = EIv$

At E, $x = \frac{1}{2}L$, and $\frac{dy}{dx} = 0$, hence

$$0 = \tfrac{1}{2}M_1 L + \frac{wL^3}{24} + EIv$$

Hence by eliminating v, we have

$$EI \frac{dy}{dx} = M_1 x - \tfrac{1}{2} M_1 L + \frac{wLx^2}{4} - \frac{wx^3}{6} - \frac{wL^3}{24}$$

or
$$EIy = \tfrac{1}{2} M_1 x^2 - \tfrac{1}{2} M_1 L x + \frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3 x}{24} + C'$$

At O, $x = 0$, and $y = 0$, hence $C' = 0$.

At E, $x = \frac{1}{2}L$, and $y = y_c$, hence $EIy_c = -\frac{1}{8} M_1 L^2 - \frac{5wL^4}{384}$

From this equation the central deflection y_c may be calculated for the several different values of the applied unbalanced moment M_1 that occur by reason of different assumed degrees of restraint at the supports, etc. These different values of M_1 as well as those of M_c may be determined when the restraints at the supports are given, for the value of M_1 will be found by the theorem of three moments and then M_c from the equation

$$M_c + M_1 = \tfrac{1}{8} wL^2$$

The results of the application of the theorem of three moments and our equation for central deflections to three pertinent cases is as follows:

Case 1. Beam fixed horizontally at both B and O

$$M_1 = -\frac{wL^2}{12}, \quad M_c = \frac{wL^2}{24}, \quad EIy_c = -\frac{wL^4}{384} = -0.0026\,wL^4.$$

Case 2. Beam fixed horizontally at end support A and the other end support but free to tip at B and O.

$$M_1 = -\frac{wL^2}{18}, \quad M_c = \frac{wL^2}{14.4}, \quad EIy_c = -\frac{7wL^4}{1152} = -0.0061\,wL^4$$

Case 3. Beam pinned but free to tip at all four supports.

$$M_1 = -\frac{wL^2}{20}, \quad M_c = \frac{3wL^2}{40}, \quad EIy_c = -\frac{13\,wL^4}{1920} = 0.0068\,wL^4.$$

It is to be noted that the deflection in the second case is two and one third times that in the first case by reason of the reduced rigidity, and by reason of the further reduction of the rigidity in the third case the deflection is increased to 2.6 times that in the first case. We may say then that lack of firm restraint by the columns at the ends of the span BO will in general more than double the central deflection. Such lack of firm restraint may be due either to lack of integral connection between columns and beam or to the small moment of inertia in the columns themselves as compared with that of the beam.

4 Unbalanced Loads on a Floor Resting on a Rectangular System of Continuous Girders Integral with its Supporting Columns. Suppose continuous beams both lengthwise and crosswise of the floor between columns, then in addition to the resistances previously considered as acting at O in a continuous beam, any applied moment M_1 will be resisted by twisting moments M_5 and M_6 in the cross beams extending horizontally away from O in each direction at right angles to AB. Each of these moments of resistance will be of the form

$$M = -vFJ/L \text{.................................} (13)$$

in which F = modulus of elasticity for shearing, J = polar moment of inertia, and L = effective length of cross beam from O to the fixed end, while the angle of twist = v, the inclination at O of the columns and longitudinal beams due to the applied moment M_1.

We shall then have by reasoning like that previously employed respecting (10)

$$M_1 = M_2 + M_3 + M_4 + M_5 + M_6 \text{..................} (14)$$

$$\frac{M_5}{M_1} = \frac{FJ_5/4EL_5}{I_2/L_2 + I_3/L_3 + I_4/L_4 + (J_5/L_5 + J_6/L_6)F/4E} \quad (15)$$

with a similar expression for M_6/M_1.

The expressions for M_3/M_1, M_2/M_1 and M_4/M_1, will have the same numerators as in (12) etc., and the same denomenator as in (15).

In case any of the members here considered are absent, the values of the moments of inertia of those members are to be placed equal to zero in all these expressions, just as has been done in the previous part of this treatment. It is to be noticed, since moments are directed quantities, that the transverse moments are independent of the lengthwise moments and that the twisting moment in any beam is independent of its bending moment.

The effect of the twisting resistance of cross beams is to increase the rigidity of the structure and to decrease the bending moments at O in the columns and in the span OA.

It should be noticed that no twisting deformation may be produced when both ends of a transverse beam are twisted thru the same angle, as is the case when a bay of panels side by side are each loaded with equal unbalanced loads. The case treated in (14) and (15) is that of an unbalanced twist. A live load placed upon a single panel of such a floor with O at the one corner of the panel would not cause twists in any of the four beams at the edges of that panel, but would cause twists in all the other eight beams radiating from the corners in case the floor is assumed to be merely supported upon them. In case the beams are integral with the floor the tipping of the floor at the panel edges will twist the beams,

5. Flat Slab Floors. The foregoing investigation may be applied with suitable modifications to the unbalanced moments in a flat slab floor integral with a rectangular array of supporting columns; as for example where there is no unbalanced load to the right of an entire transverse row of columns such as CD one behind another across the entire width of the floor while the unbalanced load at the left of this row is so situated as to make the applied bending moments equal to each other in the several panels of the bay just at the left of the row CD. In such a case, it is evident that the entire floor or either of the bays or strips of panels at right angles to this crosswise row of columns CD constitutes a continuous beam and I_2

may be taken to designate either the sum of the moments of inertia of all the columns such as *OD* in the row below the floor, and I_4 the sum of those above, in the entire width of the floor, while I_3 is the moment of inertia of a section extending the entire width of the floor just at the right of this double row of columns, or otherwise, I_2 and I_4 may be used to designate the moments of inertia of the single columns respectively of this double row, and I_3 that of a single panel width of the floor. This latter meaning will be employed in this chapter so that I_2 and I_4 will each refer to a single column below or above the slab, and I_3 to a single panel width of the floor. Equation (12) may then be applied among other cases to the unbalanced moments, per panel just at the right of a single transverse bay of panels extending across the entire width of the floor when the bay is uniformly loaded. Designate these unbalanced moments per panel as longitudinal. The transverse moments along the uniformly loaded transverse bay considered as a beam are balanced and are greater than the lateral unbalanced moments that would occur after the removal of the uniform load from any of the panels of the transverse bay.

When every panel in each transverse bay or aisle across the width of the floor parallel to the row of columns *CD* is loaded alike, altho the several transverse bays may not be loaded in the same manner, no unbalanced twists are produced about any line across the floor, but if the panels are not loaded alike in any given transverse bay then such twisting moments are called into play within the bay and adjacent bays as to reduce to some extent the longitudinal moments. In any such case equations (15) are to be applied and in them J_5/L_5 and J_6/L_6 refer to panels of the bay from which loads have been removed. In case of partial removal of panel loads such fractions only of the just mentioned quantities are to be introduced as are equal to the fractions of the panel loads removed, so that where none is removed these quantities do not appear.

There is one modification, however, that must be made in applying equations such as (15) to a continuous flat slab floor by reason of the fact that these equations as originally obtained referred to twisting moments in beams which were unrestrained at any points between their ends, so that the twisting moment M_5 or M_6 which was applied at the end *O* was transmitted laterally to the other end undiminished by any intermediate restraint or bending moments acting upon its sides. But this is not the case in a continuous floor slab, because any twist develops bending resistances near *O* across the sides of the crosswise trip which undergoes twisting, such resistances being distributed along those sides for some indefinite distance dependent upon the rigidity of the slab. The practical effect of this is to increase the resistance of the slab to bending. It thus appears that the final result in the case of the flat slab may be expressed correctly by (12) and similar equations, without the introduction of any polar moments of inertia J of the strip, provided the value of the moment of inertia I_3 be somewhat increased to include a section of somewhat more than a panel in width. For purposes of design no such modification need be made, because larger values of the resisting moment M_2 and M_3 and M_4 will be found, when the floor is so loaded that no twisting moments exist, than will be found after the removal of any part of the load. The greatest unbalanced bending moments M_2 and M_3 and M_4 occur under the same circumstances of longitudinal loading as in case of continuous beams.

It is therefore unimportant to consider the twisting moments that may occur in interior panels, excepting those that occur regularly in each panel with balanced as well as with unbalanced loads.

The manner in which that total moment along the edge of the panel is resisted depends upon the relative rigidities of the cross section at that edge. If no reinforcing steel is placed in the top of the slab near mid span, as is frequently the case, the position of the applied moment for this part of the edge produces twisting moments that transfer that portion of the applied moments toward the columns to points where the resistance of the side belts over the column heads affords the required resistance.

Deck floor slabs are also included as a special case of the foregoing theory by making $I_4=0$. In deck slabs the loss of rigidity due to absence of columns above the slab brings more of the applied moments to bear upon the remaining members in the same manner as was noted in the case of beams.

It is to be specially noted that in (12) and similar formulas the column lengths have a very important influence such that for spans of 20 feet or less and for column lengths of 10 feet or thereabouts the unbalanced moments in the lower floors of high buildings, where I_2 and I_4 are necessarily large, are largely resisted by the columns, while for deck floors expecially those upon tall columns the columns furnish comparatively little resistance to the unbalanced applied moment which last is resisted almost entirely by the slab itself with little assistance from the columns. As before stated, the unbalanced moments here considered are those arising from unbalanced loads, and are in addition to those arising from symmetrical loads or those uniformly distributed on all the panels.

6. **Wall Panels.** Any loading, whatever, even the dead weight of the floor itself, is an unbalanced load in this case and produces an applied moment at the exterior wall which can be resisted by the slab only by cooperation with the wall columns or the restraint of the walls themselves where the slab is built integrally therewith. Where the wall columns are sufficiently rigid the wall panel might be practically restrained at the wall column but in the construction of the wall panel the building of the slab into the wall or its integral connection with the wall beam renders its deflection between columns along the wall negligible. It therefore tends to induce a different curvature in wall panels from that which occurs in interior panels.

Under the principles demonstrated in Chapter XV, Sec's 2 and 3 the advantageous operation of radial and circular stresses depends upon double curvature. Now the straightening out or the elimination of the side deflections between columns along the wall tends to induce cylindrical curvature unless the difference in rigidity on the four sides of the panel is so equalized that equivalent relations are maintained. In the older type of Mushroom construction, the half head of the wall column and its radial bent out column rods are depended upon to shorten the span between the wall column and the next interior column and maintain sufficient rigidity between the wall columns and interior columns so that the operation of the slab would be practically the same as in interior panels. The elimination of the rigid frame work of the spider column head in flat slab designs often met with has produced a special weakness in the wall panels of such floors and a tendency to deform in cylindrical rather than dish shaped curvature.

The most scientific manner of treating wall panels is to increase the rigidity of the sides of the panel normal to the wall so that their deflection will be a mean between the deflection of the edge between the interior columns and that on the wall side, which is of negligible amount. This is accomplished to a large extent in the Spiral System by the diagonal bands which are supplemented by additional rods in the wall panels and by additional direct rods thereby increasing the total metal from twenty to twenty-five percent in the sides of the panel normal to the walls over that of the interior side parallel to the wall. By this means wall panels develop substantially the strength of interior panels, when the rigidity of the wall columns has been so proportioned that adequate restraint is secured.

7. Moments of Inertia of Flat Slab Floors and Supporting Columns. In attempting to calculate the numerical values of I, etc. there are several particulars in which the moments of inertia of flat slabs are not entirely uniform thruout the spans from column to column, The largest deviation from uniformity is due to the customary large increase of steel (and frequently of concrete also) at and around the column heads. It is usual to take account of this by making a suitable change in the length of the span by assuming the effective length of the span to be reduced to some 85 percent perhaps of the actual distance between column centers. Now the column length suffers a similar reduction by reason of its capital. The effect of these two simultaneous reductions upon (12) and similar equations is largely to neutralize each other, consequently this effect may be neglected without material error.

So far as the moment of inertia of the rest of the panel except the column heads is concerned it will be possible to assign an outside value to it large enough to include all the resistance it exerts by assuming that every part of the area of the panel is entirely covered on its tension side by two layers of crossed rods spaced at the same average distance between centers as those in the several belts of slab steel. If these crossed rods are sufficiently near each other to make a fine-grained reinforcement, as they are when the distance from center to center of rods is somewhat less than the thickness of the slab, then its action in conjunction with the concrete will be to afford a tensile resistance and moment of inertia regardless of the directions of the belts practically the same as that of a continuous sheet of metal of the same weight as the total weight of the crossed rods.

The evidence of the truth of the statement that a fine-grained reinforcement of belts of crossed rods in contact with each other affords approximately the same resistance to tension in all directions as a continuous sheet of metal of the same total weight is found in the recorded results of extensometer measurements of load tests, but these measurements show a still further increase in the resisting moments of the steel due to the fact that the proportion of the total bond shear that is held in equilibrium laterally by mutual coaction between the reinforcing rods is larger than that held in equilibrium otherwise.

It is to be noted in this statement that the action of the concrete by its coaction with the reinforcement is explicitly allowed for, and no direct tensile stress in it parallel with the steel need be considered. The state of stress here under consideration presupposes a maximum unit tensile stress in the steel of 5000 lbs. or more. At smaller unit stresses direct tensile stresses in the concrete will be more or less in evidence and the moments of inertia which express the moments of resistance will be larger for that reason.

A mean value of the moment of inertia of a panel width of the slab may be written in the form

$$I_3 = id^2 j \Sigma A \text{..............................} (16)$$

in which ΣA is the total cross section of both sets of crossed rods parallel to the sides.

A somewhat larger value of I_3 which may be used as an estimate of its largest possible value, which may be used for purposes of comparison, may be calculated in taking $j \Sigma A = 6A_1$ as given in (41) p. 189 of Concrete Steel Construction, edition of 1914, in which A_1 is the cross section of one side belt. Since (41) refers explicitly to the cross section at the cap the values of i, j and d will not be the same when (16) refers to this section as when it refers, as first proposed, to a mean estimated section across the panel any where. This will appear later in the numerical examples.

Let the supporting columns be of the ordinary pattern in their spiral and vertical steel reinforcements. In calculating the moment of inertia of each column two quite different cases must be distinguished from each other, viz. The first and most unusual case is where tension actually occurs under unbalanced load in the concrete in the extreme fiber of the column just below the cap. This may take place in columns supporting deck slabs, because the total compression in such columns is less than in columns having several stories above them, and where greater unbalanced moments are brought to bear upon them because of the absence of any columns above the slab. In this case of actual tension in the extreme fiber at that point of the column where the moment is greatest, the tensile resistance of the concrete must be entirely neglected, so that in calculating the moment of inertia of the column the vertical steel alone must be taken as affording whatever resistance the column as a whole offers to the unbalanced moment, altho since in fact the extreme fiber is not in tension thruout the entire length of the column there may be a somewhat greater moment of inertia operative in part of the column length than that arising from vertical steel alone. The algebraic expression for the moment of inertia of the vertical steel is

$$I_2=\tfrac{1}{2}A_2\,r_2^2 \quad\text{------------}\quad (17)$$

in which A_3 is the cross sectional area of the vertical steel, and r_3 is the distance of the axis of each vertical rod from the axis of the column. This formula assumes that the moment of inertia of the verticals is the same as tho the same amount of steel were uniformly distributed around the column in the form of a thin cylinder of radius equal to the distance of the centers of the rods from the axis of the column, an assumption that is correct for vertical rods equally spaced around the inner circumference of the spiral.

The spirals do not add to the moment of inertia because they are not under tension except to a very limited extent under ordinary test loading unless it happens that during construction the filling and removal of the column forms has been conducted in such a manner as to develop considerable hydraulic circumferential stresses in the spirals near the column heads. Where that is the case and such stresses are developed in the upper part of the columns just below the column caps the moments of inertia may be somewhat increased by the spiral steel, but the effect had best be regarded as part of that included in the reduction of length already mentioned as negligible.

It thus appears in this first case that the rigidity at O is not only reduced by the absence of any column above the slab but also by a reduction of the rigidity in the column below by reason of the ineffectiveness of the tensile resistance of concrete, a condition of things which, for a given value of M_1, abnormally increases M_3 and v, a result of much importance in the design of deck slabs for bridges, etc., which is further complicated by the fact that these structures are usually entirely made up of outside or exterior panels in which as we have already pointed out every load whatsoever gives rise to unbalanced moments and to no others.

The second case is the one usually occurring in practice where the column is in compression thruout every cross section of its entire length, and the effect of the unbalanced moment is to increase to some extent the compression at one side of the column and decrease it at the other. In this case the compressions in the concrete act in parallel with those in the vertical steel, while the effect of the spirals may be neglected as in the first case. Hence the moment of resistance of the column is the sum total of that of the vertical steel and the concrete added together, the steel being reckoned as was done in the first case and the resistance of the concrete is that due to the full outside diameter of the column, and not merely that due to the core of the column inside the spiral which alone must be considered in testing columns to destruction.

In this case the value of I_2 to be employed in equations (12), (15) etc., will consist of two parts as follows:

$$I_2 = \tfrac{1}{2} A_s r_s^2 + \tfrac{1}{4} A_c r_c^2 / n \qquad (18)$$

in which the first part for the steel is the same as (17), and the second part for the concrete is the product of A_c the actual cross section of the concrete acting by $\frac{1}{4} r_c^2$ the square of the radius of gyration of the circular cross section of the concrete having a radius r_c, this product being reduced to a parity with steel by dividing it by $n = E_s / E_c$ as is necessary to be done in order to use it in conjunction with any moment of inertia of steel sections.

8. Unbalanced Test Loads and Modification of Slab Steel Stresses by Column Bending in Column Supported Slabs. Test loads applied to floors are usually double the rated working load or sometimes twice the working load plus the amount of the dead load all applied over one or more panels. The load is commonly placed on four panels about a column and then doubled up over two of them, these two being sometimes adjacent to each other side by side and sometimes adjacent to each other diagonally. These various cases will accordingly be compared as to the amount of the unbalanced moments they bring to bear upon the columns at the edge of the load and at mid span.

I. In a single panel test the number of supporting columns is four. One quarter of the panel load is an unbalanced load with respect to each of the four supporting columns and the tendency of each of the columns is to bend toward the center of gravity of the load.

II. When four adjacent panels are loaded, there are nine columns all told which support the test load. There is no unbalanced load, whatever, about the central column. Each of the four columns at the middle of each side has approximately half a panel of unbalanced load and each corner column has an unbalanced load of one quarter of a panel.

III. When the load of Case II over four panels is doubled up on diagonally adjacent panels there are seven supporting columns and no unbalanced load is brought upon the center column. Approximately one quarter of a panel of unbalanced load is brought on each of the other six supporting columns.

IV. When the test load covers two adjacent panels side by side there are six supporting columns and the load is divided approximately in such wise that the unbalanced load on the two central columns is a half panel load, and on each of the other four a quarter of a panel load only. This division being approximately exact for fairly stiff columns.

Any unbalanced load causes an amount of bending in the columns that is inversely proportional to the stiffness of the columns which last is proportional to the moment of inertia of the columns. Thus in a column heavily loaded by other than the unbalanced panel load the concrete may resist through its entire section by a mere shift (due to the bending) of the intensity of the uniform compression. Where, however, the load, other than the unbalanced load is so small that the bending brings tensions upon part of the steel in the cross section, the moment of inertia of the column is reduced to that of the steel on the tension side and the stiffness of the column is reduced in proportion to the reduction of the moment of inertia. Any such reduction of the moment of inertia increases the angle v.

It is perfectly possible to apply the methods of analysis that have been employed in this chapter to each and all of these cases to derive exact solutions for them. The following method, however, will bring out points of importance with sufficient accuracy to permit of the comparison of the effects of the several cases of test loading that have been enumerated.

In case of an unbalanced moment at the edge of a load such as arises in cases I to IV where the moments of inertia of the supporting columns are so large in proportion to that of the panel itself as to afford nearly perfect restraint at the edge of the load, it is possible to make a close estimate of the relative magnitudes of the unbalanced moments produced at mid span in the several cases previously described. The magnitudes of these unbalanced moments at mid span are independent of the magnitude of the total moments at mid span, and refer only to the corrections that are to be made to the values they would have were all panels uniformly loaded, by reason of the load being confined to one, two or four panels as described in Cases I to IV instead of being uniformly distributed on all panels. At the edge of a load next to a column the difference between the moment that actually exists and that which would exist were all panels equally loaded is the unbalanced moment and it changes the slope of the slab and column by some angle e which is directly proportional to the magnitude of the unbalanced moment, e being the direct effect and measure of its action.

A change of slope at a column at one edge of a test load which is due to an unbalanced moment at the point will produce a change of slope throut the span which will gradually diminish in amount to the opposite column if there is none at that column. The change of slope will diminish uniformly to the column opposite and the change of slope at mid span will be half that at the edge, if we assume that the moment of inertia is practically constant thruout the span. Hence the unbalanced moment at mid span will be half that introduced at the edge.

In case II where four adjacent panels are loaded the unbalanced load will have approximately twice as great an effect to bend each of the four columns at the middles of the sides toward the center column as it will have to bend the corner columns in the same directions. But each corner column being subjected simultaneously to two flexures at right angles to each other, will have a resultant bending toward the center column such that if e designate the bending of a column at the middle of a side and $\frac{1}{2}e$ that component of the bending of a corner column that is parallel to it, then $e/\sqrt{2}$ is the resultant bending toward the center column of a corner column.

In Case III when half the loading of Case II has been removed by taking the load from two diagonal panels of the four loaded in Case II it is evident that each side column in Case II owes one half of its bending e to one pair of diagonal panel loads and one half to the other pair. It is therefore clear that in Case III each of the seven supporting columns except the center one is bent approximately in the direction of each of the sides by an amount $= \frac{1}{2}e$ and to a resultant amount $= e/\sqrt{2}$ toward the center of the loaded panel it supports.

In Case I the same may be taken as true respecting each of the four supporting columns but in this case the opposite slopes at mid span cancel each other so that no unbalanced moment exists in this case at mid span. Nevertheless, the resultant bending moment at mid span has been increased by half the sum of the unbalanced bending moments at the edges of the panel.

In Case IV of two adjacent panels on six columns the bending at the four corner columns is approximately $\frac{1}{2}e$ in each direction, but at the two middle columns it is e towards each other from which results the unbalanced moments at mid span and edges of load are seen in all these cases to be measured either by e or $\frac{1}{2}e$ in the direction of the sides.

ADVANCE NOTICE

SECTION IV WILL BE DEVOTED TO ANALYSIS OF COLUMNS, COMPRESSION MEMBERS OR STRUTS OF REINFORCED CONCRETE AND STRUCTURAL STEEL, DISCUSSION OF FASTENINGS AND SKEW DETAILS, ROOF AND BRIDGE WORK.

SECTION V WILL DEAL WITH ECONOMIC THEORY OF TRIANGULATION OF FRAME STRUCTURES.

TA
405
T8
v.3

Turner, Claude Allen Porter
Elasticity and strength
of materials in construction

Engineering

PLEASE DO NOT REMOVE
CARDS OR SLIPS FROM THIS POCKET

UNIVERSITY OF TORONTO LIBRARY

[redacted]